



DEPARTMENT OF MATHEMATICS

UNIT I - PARTIAL DIFFERENTIAL EQUATIONS

1.1 FORMATION OF PDE BY ELIMINATING ARBITRARY CONSTANTS & FUNCTIONS

Notations: If $z = f(x, y)$ then

$$p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}; r = \frac{\partial^2 z}{\partial x^2}; s = \frac{\partial^2 z}{\partial x \partial y}; t = \frac{\partial^2 z}{\partial y^2}$$

Formation of PDE by eliminating arbitrary constants:

Let the given equation be $z = f(x, y, a, b)$ ----- (1)

Step 1: Differentiating (1) partially with respect to x

$$\frac{\partial z}{\partial x} = p = f'(x, y, a, b) \text{ ----- (2)}$$

Step 2: Differentiating (1) partially with respect to y

$$\frac{\partial z}{\partial y} = q = f'(x, y, a, b) \text{ ----- (3)}$$

Step 3: Eliminate a & b from (1) using (2) & (3)

1. **Obtain partial differential equation by eliminating arbitrary constant 'a' and 'b' from**

$$z = (x-a)^2 + (y-b)^2$$

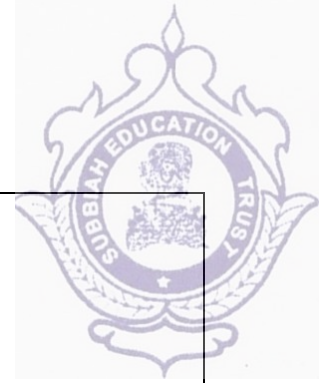
Solution:

Given $z = (x-a)^2 + (y-b)^2$ ----- (1)

Diff Partially w.r.t x

$$\frac{\partial z}{\partial x} = 2(x-a) + 0$$

$$p = 2(x-a) \text{ ----- (2)}$$



Diff Partially w.r.t y

$$\frac{\partial z}{\partial y} = 0 + 2(y - b)$$

$$q = 2(y - b) \text{ ----- (3)}$$

Eliminate a & b from (1) using (2) & (3)

$$(2) \Rightarrow (x - a) = \frac{p}{2} \text{ ----- (4)}$$

$$(3) \Rightarrow y - b = \frac{q}{2} \text{ ----- (5)}$$

Sub (4) & (5) in (1)

$$(1) \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

The required the PDE is

$$\boxed{p^2 + q^2 = 4z}$$

2. **Form the partial differential equation by eliminating the arbitrary constants 'a' & 'b' from**

$$z = (x^2 + a)(y^2 + b).$$

Solution:

$$\text{Given } z = (x^2 + a)(y^2 + b) \text{ ----- (1)}$$

Diff Partially w.r.t x

$$\frac{\partial z}{\partial x} = p = 2x(y^2 + b) \text{ ----- (2)}$$

Diff Partially w.r.t y

$$\frac{\partial z}{\partial y} = q = 2y(x^2 + a) \text{ ----- (3)}$$

Eliminate a & b from (1) using (2) & (3)

$$(2) \Rightarrow (y^2 + b) = \frac{p}{2x} \text{ ----- (4)}$$



$$(3) \Rightarrow x^2 + b = \frac{q}{2y} \text{-----}(5)$$

Sub (4) & (5) in (1)

$$(1) \Rightarrow z = \left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right)$$

The required the PDE is

$$\boxed{4xyz = pq}$$

3. **Find the PDE of all planes having equal intercepts on the x and y axis.**

Solution:

The intercept form of the plane equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Given that equal intercepts on the x & y axis $\Rightarrow a = b$

$$\therefore \frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1 \text{-----}(1)$$

Diff Partially w.r.t x

$$\frac{1}{a} + 0 + \frac{1}{c} \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{1}{a} = \frac{-1}{c} p \text{-----}(2)$$

Diff Partially w.r.t y

$$0 + \frac{1}{a} + \frac{1}{c} \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{1}{a} = \frac{-1}{c} q \text{-----}(3)$$

From (2) & (3) $\frac{-1}{c} p = \frac{-1}{c} q$ The required the PDE is $\boxed{p = q}$

4. **Obtain the partial differential equation by eliminating arbitrary constants 'a' and 'b' from**

$$(x-a)^2 + (y-b)^2 + z^2 = r^2$$

Solution:

$$(x-a)^2 + (y-b)^2 + z^2 = 1 \text{-----}(1) \quad - (1)$$

Diff Partially w.r.t x



$$2(x-a)(1-0) + 0 + 2z \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow 2(x-a) + 2zp = 0 \text{ ----- (2)}$$

Diff Partially w.r.t y

$$0 + 2(y-b)(1-0) + 2z \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow 2(y-b) + 2zq = 0 \text{ ----- (3)}$$

Eliminate a & b from (1) using (2) & (3)

$$(2) \Rightarrow x-a = -zp \text{ ----- (4)}$$

$$(3) \Rightarrow y-b = -zq \text{ ----- (5)}$$

Sub (4) & (5) in (1)

$$(-zp)^2 + (-zq)^2 + z^2 = 1$$

The required PDE is $z^2(p^2 + q^2 + 1) = 1$

Formation of PDE by eliminating arbitrary functions:

1. **Eliminate the arbitrary function f from $z = f\left(\frac{y}{x}\right)$ and form the PDE.**

Solution:

$$z = f\left(\frac{y}{x}\right) \text{ ----- (1)}$$

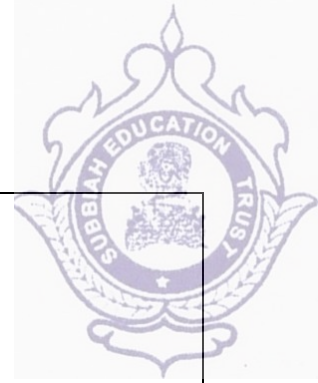
Diff Partially w.r.t x

$$\frac{\partial z}{\partial x} = p = f'\left(\frac{y}{x}\right) \times \left(\frac{-y}{x^2}\right) \Rightarrow f'\left(\frac{y}{x}\right) = \frac{-px^2}{y} \text{ ----- (2)}$$

Diff Partially w.r.t y

$$\frac{\partial z}{\partial y} = q = f'\left(\frac{y}{x}\right) \times \left(\frac{1}{x}\right) \text{ ----- (3)}$$

From (1) & (2) $\frac{p}{q} = \frac{-y}{x} \Rightarrow \boxed{px + qy = 0}$



2.

Form the partial differential equation by eliminating f from $z = x^2 + 2f\left(\frac{1}{y} + \log x\right)$.

Solution:

Given $z = x^2 + 2f\left(\frac{1}{y} + \log x\right)$ ----- (1)

Differentiate (1) partially w.r. t x

$$\frac{\partial z}{\partial x} = 2x + 2f'\left(\frac{1}{y} + \log x\right)\left(0 + \frac{1}{x}\right)$$

$$p = 2x + 2f'\left(\frac{1}{y} + \log x\right)\left(\frac{1}{x}\right) \Rightarrow f'\left(\frac{1}{y} + \log x\right) = (p - 2x)\frac{x}{2}$$
 ----- (2)

$$\frac{\partial z}{\partial y} = 2f'\left(\frac{1}{y} + \log x\right)\left(\frac{-1}{y^2} + 0\right)$$

$$q = \frac{-2}{y^2} f'\left(\frac{1}{y} + \log x\right) \Rightarrow f'\left(\frac{1}{y} + \log x\right) = \frac{-qy^2}{2}$$
 ----- (3)

Eliminating f' from (2) & (3)

$$(p - 2x)\frac{x}{2} = \frac{-qy^2}{2} \Rightarrow (px - 2x^2) = -qy^2$$

$$\Rightarrow \boxed{px + qy^2 = 2x^2}$$

Formation of PDE by eliminating f from $f(u, v) = 0$ ----- (1)

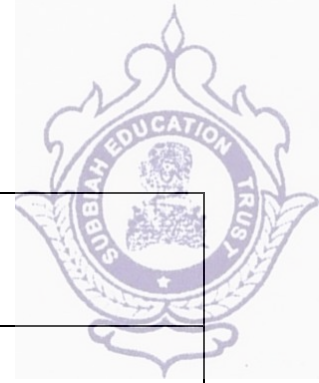
Method 1:

The required PDE of (1) is
$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

Method 2:

The required PDE is $Pp + Qq = R$

Where



$$P = \begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix}; Q = \begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix}; R = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}$$

1. **Form the PDE from $\phi(ax+by+cz, x^2+y^2+z^2)=0$**

Solution:

Given $\phi(ax+by+cz, x^2+y^2+z^2)=0$

This is of the form $f(u,v)=0$ where $u = ax+by+cz$ & $v = x^2+y^2+z^2$

The required PDE of (1) is
$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ a & b & c \\ 2x & 2y & 2z \end{vmatrix} = 0$$

$$\Rightarrow p(2bz - 2cy) - q(2az - 2cx) + 1(2az - 2cx) = 0$$

$$\div 2 \Rightarrow \boxed{(bz - cy)p + (cx - az)q + (az - cx) = 0}$$

2. **Form the PDE from $\phi(x^2+y^2+z^2, xyz)=0$**

Solution:

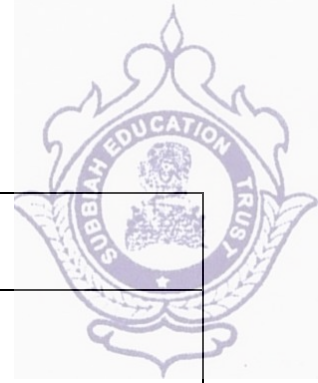
Given $\phi(x^2+y^2+z^2, xyz)=0$

This is of the form $f(u,v)=0$ where $u = x^2+y^2+z^2$ & $v = xyz$

The required PDE of (1) is
$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ 2x & 2y & 2z \\ yz & xz & xy \end{vmatrix} = 0$$

$$\Rightarrow p(2xy^2 - 2xz^2) - q(2x^2y - 2yz^2) + 1(2x^2z - 2y^2z) = 0$$



$$\div 2 \Rightarrow x(y^2 - z^2)p + y(z^2 - x^2) + z(x^2 - y^2) = 0$$

3. Form the PDE from $\phi\left(\frac{y}{x}, x^2 + y^2 + z^2\right) = 0$

Solution:

$$\text{Given } \phi\left(\frac{y}{x}, x^2 + y^2 + z^2\right) = 0$$

This is of the form $f(u, v) = 0$ where $u = \frac{y}{x}$ & $v = x^2 + y^2 + z^2$

$$\text{The required PDE of (1) is } \begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ \frac{-y}{x^2} & \frac{1}{x} & 0 \\ 2x & 2y & 2z \end{vmatrix} = 0$$

$$\Rightarrow p\left(\frac{2z}{x} - 0\right) - q\left(\frac{-2yz}{x^2} - 0\right) + 1\left(\frac{-2y^2}{x^2} - \frac{2x}{x}\right) = 0$$

$$\Rightarrow \frac{2zp}{x} + \frac{2yzq}{x^2} - 2\left(\frac{y^2}{x^2} + 1\right) = 0$$

$$\div 2 \Rightarrow \frac{zp}{x} + \frac{yzq}{x^2} - \left(\frac{y^2 + x^2}{x^2}\right) = 0$$

$$\Rightarrow \frac{xzp + yzq - (y^2 + x^2)}{x^2} = 0$$

$$\boxed{xzp + yzq - (y^2 + x^2) = 0}$$



DEPARTMENT OF MATHEMATICS

UNIT I - PARTIAL DIFFERENTIAL EQUATIONS

1.2 SOLUTIONS OF STANDARD TYPES OF FIRST ORDER PDE

Solutions of standard types of First order PDE's:

Different solutions of PDE:

Complete Integral (or) Complete Solution:

If the number of arbitrary constants is equal to number of independent variables, then the solution is called Complete integral.

Singular Integral (or) Singular Solution:

Consider a PDE of first order as $f(x, y, z, p, q) = 0$ -----(1)

It's complete integral may be, $f(x, y, z, a, b) = 0$ -----(2)

Diff (2) partially with respect to a & b respectively,

$$\frac{\partial g}{\partial a} = 0 \text{ -----(3)}$$

$$\frac{\partial g}{\partial b} = 0 \text{ -----(4)}$$

Eliminating a & b from (3) & (4) will get the Singular integral.

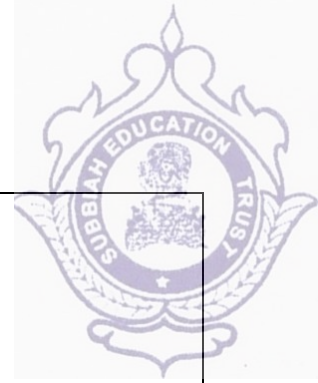
General Integral (or) Complete solution:

A Solution which contains number of arbitrary functions is equal to the order of the given PDE.

(or) A solution which contains the maximum possible number of arbitrary functions.

Type I:

Equations of the form $f(p, q) = 0$ -----(1)

**To find Complete Integral:**

Let the complete solution of (1) is $z = ax + by + c$ -----(2)

Let $p = a$ & $q = b$ in (1)

$f(a, b) = 0$ and represent $b = \phi(a)$

$\therefore (1) \Rightarrow z = ax + \phi(a)y + c$ -----(3)

To Find Singular Integral:

Diff (3) partially with respect to c

$0 = 1$ which is impossible

There is no singular integral for this type.

To find General integral:

Put $c = g(a)$ in (3)

(3) $\Rightarrow z = ax + \phi(a)y + g(a)$ -----(4)

Diff (4) partially with respect to a

$0 = x(1) + \phi'(a)y + g'(a)$ -----(5)

Eliminating a from (4) & (5) we get general integral.

1. **Find the complete integral of $p + q = pq$**

Solution:

Given $p + q = pq$ -----(1)

This of the form $f(p, q) = 0$

To find Complete Integral:

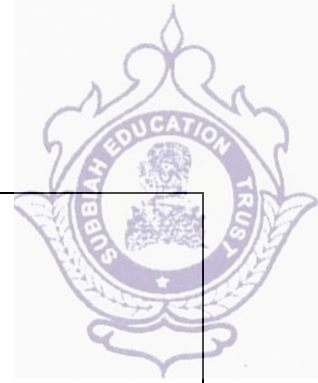
Let the complete solution of (1) is $z = ax + by + c$ -----(2)

Let $p = a$ & $q = b$ in (1)

(1) $\Rightarrow a + b = ab \Rightarrow a + b - ab = 0 \Rightarrow b = \frac{a}{a-1}$



	<p>Sub b in (2)</p> $z = ax + \left(\frac{a}{a-1}\right)y + c$ <p>This is the required complete integral.</p>
2.	<p>Find the complete integral of $p + q = 1$</p> <p>Solution:</p> <p>Given $p + q = 1$ -----(1)</p> <p>This of the form $f(p, q) = 0$</p> <p>To find Complete Integral:</p> <p>Let the complete solution of (1) is $z = ax + by + c$ -----(2)</p> <p>Let $p = a$ & $q = b$ in (1)</p> <p>(1) $\Rightarrow a + b = 1 \Rightarrow b = 1 - a$</p> <p>Sub b in (2)</p> $z = ax + (1 - a)y + c$ <p>This is the required complete integral.</p>
3.	<p>Solve $\sqrt{p} + \sqrt{q} = 1$</p> <p>Solution:</p> <p>Given $\sqrt{p} + \sqrt{q} = 1$ -----(1)</p> <p>This of the form $f(p, q) = 0$</p> <p>To find Complete Integral:</p> <p>Let the complete solution of (1) is $z = ax + by + c$ -----(2)</p> <p>Let $p = a$ & $q = b$ in (1)</p> <p>(1) $\Rightarrow \sqrt{a} + \sqrt{b} = 1 \Rightarrow \sqrt{b} = 1 - \sqrt{a} \Rightarrow b = (1 - \sqrt{a})^2$</p>



Sub b in (2)

$$z = ax + (1 - \sqrt{a})^2 y + c \quad \text{-----(3)}$$

This is the required complete integral.

To Find Singular Integral:

Diff (3) partially with respect to c

$$0 = 1 \text{ which is impossible}$$

There is no singular integral for this type.

To find General integral:

Put $c = f(a)$ in (3)

$$(3) \Rightarrow z = ax + (1 - \sqrt{a})^2 y + f(a) \quad \text{-----(4)}$$

Diff. (4) partially with respect to a

$$0 = x(1) + 2(1 - \sqrt{a}) \frac{1}{2\sqrt{a}} y + f'(a) \quad \text{-----(5)}$$

Eliminate a from (4) & (5) we get the general integral.

Type II:

Equations of the form $z = px + qy + f(p, q) \quad \text{-----(1)}$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$$\therefore (1) \Rightarrow z = ax + by + f(a, b) \quad \text{-----(2)}$$

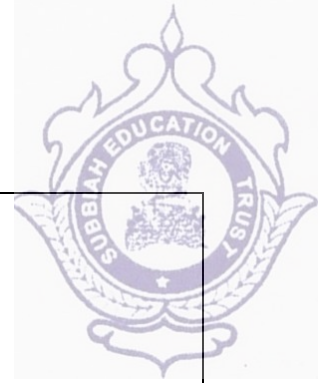
To Find Singular Integral:

Diff (2) partially with respect to a

$$0 = x(1) + 0 + f'(a, b) \quad \text{-----(3)}$$

Diff (2) partially with respect to b

$$0 = 0 + y(1) + f'(a, b) \quad \text{-----(4)}$$



Eliminating a & b from (2) using (3) & (4), we get the singular integral.

To find General integral:

Put $b = \phi(a)$ in (2)

$$(2) \Rightarrow z = ax + \phi(a)y + g(a) \text{ -----(5)}$$

Diff (4) partially with respect to a

$$0 = x(1) + \phi'(a)y + g'(a) \text{ -----(6)}$$

Eliminating a from (5) & (6) we get General integral.

1. **Solve** $z = px + qy + p^2q^2$

Solution:

$$\text{Given } z = px + qy + p^2q^2 \text{ -----(1)}$$

Equations of the form $z = px + qy + f(p, q)$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$$\therefore (1) \Rightarrow \boxed{z = ax + by + a^2b^2} \text{ -----(2)}$$

This is the required complete integral

To Find Singular Integral:

Diff (2) partially with respect to a

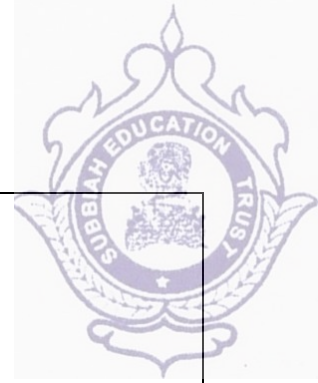
$$0 = x(1) + 0 + 2ab^2 \Rightarrow x + 2ab^2 = 0 \Rightarrow x = -2ab^2 \text{ -----(3)}$$

Diff (2) partially with respect to b

$$0 = 0 + y(1) + 0 + 2a^2b \Rightarrow y + 2a^2b = 0 \Rightarrow y = -2a^2b \text{ -----(4)}$$

Eliminating a & b from (2) using (3) & (4)

$$(3) \Rightarrow \frac{x}{b} = -2ab \text{ -----(5)}$$



$$(4) \Rightarrow \frac{y}{a} = -2ab \text{ -----(6)}$$

From (5) & (6)

$$\frac{x}{b} = \frac{y}{a} = k \text{ (say)}$$

$$\Rightarrow \frac{x}{b} = k \text{ \& \ } \frac{y}{a} = k$$

$$\Rightarrow b = \frac{x}{k} \text{ \& \ } a = \frac{y}{k} \text{ -----(7)}$$

Sub a & b in (2)

$$(2) \Rightarrow z = \frac{y}{k}x + \frac{x}{k}y + \frac{y^2}{k^2} \frac{x^2}{k^2}$$

$$z = \frac{xy}{k} + \frac{xy}{k} + \frac{x^2y^2}{k^4}$$

$$z = \frac{2xy}{k} + \frac{x^2y^2}{k^4} \text{ -----(8)}$$

To find k

Sub (7) in (3) (or) (4)

$$(3) \Rightarrow x = -2 \frac{y}{k} \frac{x^2}{k^2} \Rightarrow x = \frac{-2x^2y}{k^3} \Rightarrow k^3 = -2xy$$

$$(8) \Rightarrow z = \frac{2xy}{k} + \frac{x^2y^2}{k(-2xy)} \Rightarrow z = \frac{2xy}{k} - \frac{xy}{2k}$$

$$z = \frac{4xy - xy}{2k} \Rightarrow z = \frac{3xy}{2k} \Rightarrow z^3 = \frac{27x^3y^3}{8k^3} \Rightarrow z^3 = \frac{27x^3y^3}{8(-2xy)}$$

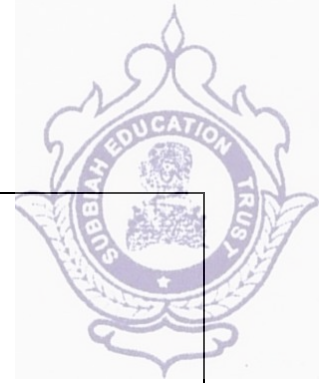
$$\boxed{16z^3 = -27x^2y^2}$$

This is the required singular integral.

To find General integral:



	<p>Put $b = \phi(a)$ in (2)</p> <p>(2) $\Rightarrow z = ax + f(a)y + a^2 [f(a)]^2$ -----(9)</p> <p>Diff (9) partially with respect to a</p> <p>$0 = x(1) + f'(a)y + 2f(a)f'(a)$ -----(10)</p> <p>Eliminating a from (9) & (10) we get General integral.</p>
2.	<p>Find the singular integral of $z = px + qy + p^2 + pq + q^2$</p> <p>Solution:</p> <p>Given $z = px + qy + p^2 + pq + q^2$ ----- (1)</p> <p>Equations of the form $z = px + qy + f(p, q)$</p> <p>To find Complete Integral:</p> <p>Put $p = a$ & $q = b$ in (1)</p> <p>\therefore (1) \Rightarrow $z = ax + by + a^2 + ab + b^2$ -----(2)</p> <p>This is the required complete integral</p> <p>To Find Singular Integral:</p> <p>Diff (2) partially with respect to a</p> <p>$0 = x(1) + 0 + 2a + b + 0 \Rightarrow 2a + b = -x$ -----(3)</p> <p>Diff (2) partially with respect to b</p> <p>$0 = 0 + y(1) + 0 + a + 2b \Rightarrow a + 2b = -y$ -----(4)</p> <p>Eliminating a & b from (2) using (3) & (4)</p> <p>(4) $\times 2 \Rightarrow 2a + 4b = -2y$ -----(5)</p> <p>(3) - (5) $\Rightarrow -3b = -x + 2y \Rightarrow b = \frac{x - 2y}{3}$</p> <p>Sub the value of b in (3)</p>



$$2a + b = -x \Rightarrow 2a = -x - b \Rightarrow 2a = -x - \left(\frac{x-2y}{3}\right)$$

$$2a = \frac{-3x - x + 2y}{3} \Rightarrow 6a = -4x + 2y \Rightarrow a = \frac{y-2x}{3}$$

Sub the value of a & b in (2)

$$z = \left(\frac{y-2x}{3}\right)x + \left(\frac{x-2y}{3}\right)y + \left(\frac{y-2x}{3}\right)^2 + \left(\frac{y-2x}{3}\right)\left(\frac{x-2y}{3}\right) + \left(\frac{x-2y}{3}\right)^2$$

$$z = \frac{xy - 2x^2}{3} + \frac{xy - 2y^2}{3} + \frac{y^2 - 4xy + 4y^2}{9} + \frac{xy - 2y^2 - 2x^2 + 4xy}{9} + \frac{x^2 - 4xy + 4y^2}{9}$$

$$z = \frac{3xy - 3x^2 + 3xy - 6y^2 + y^2 - 4xy + 4y^2 + xy - 2y^2 - 2x^2 + 4xy + x^2 - 4xy + 4y^2}{9}$$

$$\boxed{9z = -4x^2 + y^2 + xy}$$

3.

Solve $z = px + qy + \sqrt{p^2 + q^2 + 1}$

Solution:

Given

$$z = px + qy + \sqrt{p^2 + q^2 + 1} \text{ ----- (1)}$$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$$(1) \Rightarrow \boxed{z = ax + by + \sqrt{a^2 + b^2 + 1}} \text{ ----- (2)}$$

This is required complete integral.

To Find Singular Integral:

Diff (2) partially with respect to a

$$0 = x(1) + 0 + \frac{1}{2\sqrt{a^2 + b^2 + 1}} (2a) \Rightarrow \boxed{x = \frac{-a}{\sqrt{a^2 + b^2 + 1}}} \text{ ----- (3)}$$



Diff (2) partially with respect to b

$$0 = 0 + y(1) + \frac{1}{2\sqrt{a^2 + b^2 + 1}}(2b) \Rightarrow y = \frac{-b}{\sqrt{a^2 + b^2 + 1}} \text{-----(4)}$$

Eliminating a & b from (2) using (3) & (4)

$$(3)^2 + (4)^2 \Rightarrow x^2 + y^2 = \left(\frac{-a}{\sqrt{a^2 + b^2 + 1}} \right)^2 + \left(\frac{-b}{\sqrt{a^2 + b^2 + 1}} \right)^2$$

$$x^2 + y^2 = \frac{a^2}{a^2 + b^2 + 1} + \frac{b^2}{a^2 + b^2 + 1}$$

$$x^2 + y^2 = \frac{a^2 + b^2}{a^2 + b^2 + 1}$$

$$1 - (x^2 + y^2) = 1 - \frac{a^2 + b^2}{a^2 + b^2 + 1}$$

$$1 - x^2 - y^2 = \frac{a^2 + b^2 + 1 - a^2 - b^2}{a^2 + b^2 + 1}$$

$$1 - x^2 - y^2 = \frac{1}{a^2 + b^2 + 1}$$

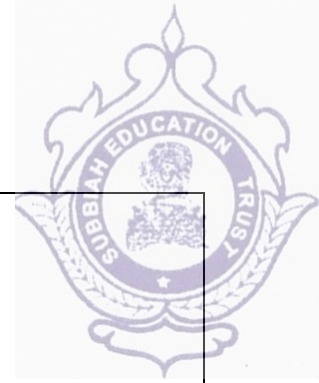
$$\Rightarrow 1 + a^2 + b^2 = \frac{1}{1 - x^2 - y^2}$$

Taking square root on both sides

$$\Rightarrow \sqrt{1 + a^2 + b^2} = \frac{1}{\sqrt{1 - x^2 - y^2}} \text{-----(5)}$$

Sub (5) in (2) and (3)

$$(3) \Rightarrow x = \frac{-a}{\frac{1}{\sqrt{1 - x^2 - y^2}}} \Rightarrow x = -a\sqrt{1 - x^2 - y^2} \Rightarrow a = \frac{-x}{\sqrt{1 - x^2 - y^2}}$$



$$(4) \Rightarrow y = \frac{-b}{\sqrt{1-x^2-y^2}} \Rightarrow y = -b\sqrt{1-x^2-y^2} \Rightarrow b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

Sub (5), a & b in (2)

$$(2) \Rightarrow z = \left(\frac{-x}{\sqrt{1-x^2-y^2}} \right) x + \left(\frac{-y}{\sqrt{1-x^2-y^2}} \right) y + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$\Rightarrow z = \frac{-x^2}{\sqrt{1-x^2-y^2}} + \frac{-y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$\Rightarrow z = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$\Rightarrow z = \sqrt{1-x^2-y^2}$$

Squaring on both sides

$$z^2 = 1-x^2-y^2 \Rightarrow \boxed{x^2+y^2+z^2=1}$$

4. **Find the singular integral of $z = px + qy + p^2 - q^2$**

Solution:

Given $z = px + qy + p^2 - q^2$ ----- (1)

Equations of the form $z = px + qy + f(p, q)$

To find Complete Integral:

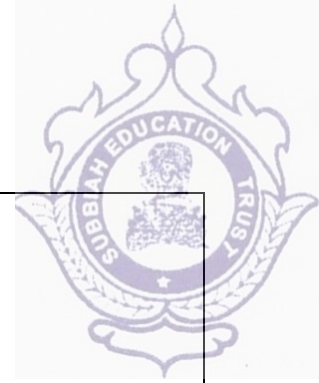
Put $p = a$ & $q = b$ in (1)

$$\therefore (1) \Rightarrow \boxed{z = ax + by + a^2 - b^2}$$
 ----- (2)

This is the required complete integral

To Find Singular Integral:

Diff (2) partially with respect to a



$$0 = x(1) + 0 + 2a + 0 \Rightarrow a = \frac{-x}{2} \text{ -----(3)}$$

Diff (2) partially with respect to b

$$0 = 0 + y(1) + 0 - 2b \Rightarrow b = \frac{y}{2} \text{ -----(4)}$$

Sub a & b in (2)

$$\therefore (2) \Rightarrow z = \left(\frac{-x}{2}\right)x + \left(\frac{y}{2}\right)y + \left(\frac{-x}{2}\right)^2 - \left(\frac{y}{2}\right)^2$$

$$\therefore (2) \Rightarrow z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

$$\therefore (2) \Rightarrow z = \frac{-2x^2 + 2y^2 + x^2 - y^2}{4} \Rightarrow \boxed{4z = -x^2 + y^2}$$

This is the required singular integral.

Type III:

Equations of the form $f(z, p, q) = 0$ ----- (1)

In this type x & y do not appear explicitly.

To find Complete Integral:

Let the complete solution of (1) is $z = f(x + ay)$ ----- (2)

Let $x + ay = u$

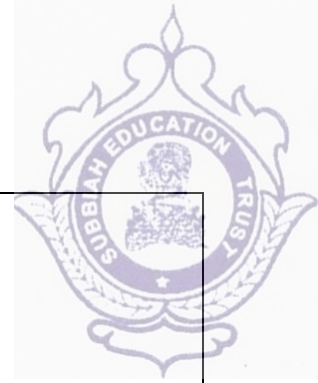
$$(2) \Rightarrow z = f(u) \text{ -----(3)}$$

By total derivative,

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \Rightarrow p = \frac{dz}{du} (1) \quad \because u = x + ay \Rightarrow \frac{\partial u}{\partial x} = 1$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} \Rightarrow q = a \frac{dz}{du} \quad \because u = x + ay \Rightarrow \frac{\partial u}{\partial y} = a$$

Substitute the value of p & q in (1)



$$(1) \Rightarrow f\left(z, \frac{dz}{du}, a \frac{dz}{du}\right) = 0$$

This may be solve by method of separation of variables

Other solutions can obtain as usual.

1. Solve $p(1+q) = qz$.

Solution:

$$\text{Given } p(1+q) = qz \text{ -----(1)}$$

This is of the form $f(z, p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = f(x+ay) \text{ -----(2)}$

$$\text{Let } x+ay = u \Rightarrow z = f(u)$$

$$\text{Then } p = \frac{dz}{du} \text{ \& } q = a \frac{dz}{du}$$

Substitute the value of p & q in (1)

$$(1) \Rightarrow \frac{dz}{du} \left(1 + a \frac{dz}{du}\right) = a \frac{dz}{du} z$$

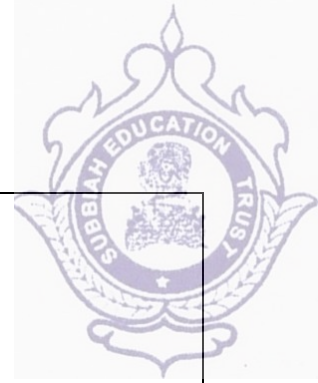
$$1 + a \frac{dz}{du} = a z$$

$$\frac{dz}{du} = a z - 1$$

$$\frac{dz}{a z - 1} = du$$

Integrating on both sides

$$\int \frac{dz}{a z - 1} = \int du$$



$$u = \log(az-1) + c \quad \because \int \frac{f'(x)}{f(x)} dx = \log f(x)$$

$$\boxed{x + ay = \log(az-1) + c}$$

This is the required complete integral.

Other solutions can be obtained as usual.

2. Solve $z^2 = 1 + p^2 + q^2$

Solution:

Given $z^2 = 1 + p^2 + q^2$ -----(1)

This is of the form $f(z, p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = f(x + ay)$ -----(2)

Let $x + ay = u \Rightarrow z = f(u)$

Then $p = \frac{dz}{du}$ & $q = a \frac{dz}{du}$

Substitute the value of p & q in (1)

$$(1) \Rightarrow z^2 = \left(\frac{dz}{du}\right)^2 + \left(a \frac{dz}{du}\right)^2 + 1$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 + a^2 \left(\frac{dz}{du}\right)^2 = z^2 - 1$$

$$\Rightarrow (1 + a^2) \left(\frac{dz}{du}\right)^2 = z^2 - 1$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{z^2 - 1}{1 + a^2}$$

Taking square root on both sides



$$\frac{dz}{du} = \frac{\sqrt{z^2 - 1}}{\sqrt{1 + a^2}}$$

$$\Rightarrow \frac{dz}{\sqrt{z^2 - 1}} = \frac{du}{\sqrt{1 + a^2}}$$

Integrating on both sides

$$\cosh^{-1} z = \frac{1}{\sqrt{a^2 - 1}} u + c \quad \because \int \frac{dz}{\sqrt{x^2 - 1}} = \cosh^{-1} x$$

$$\boxed{\cosh^{-1} z = \frac{1}{\sqrt{a^2 - 1}} (x + ay) + c} \quad \because u = x + ay$$

This is the required complete integral.

Other solutions can be obtained as usual.

3.

Solve $p(1 - q^2) = q(1 - z)$

Solution:

Given $p(1 - q^2) = q(1 - z)$ -----(1)

This is of the form $f(z, p, q) = 0$

To find Complete Integral:

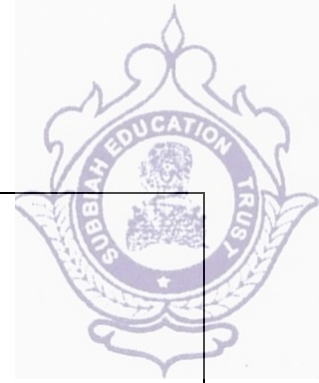
Let the complete solution of (1) is $z = f(x + ay)$ -----(2)

Let $x + ay = u \Rightarrow z = f(u)$

Then $p = \frac{dz}{du}$ & $q = a \frac{dz}{du}$

Substitute the value of p & q in (1)

$$(1) \Rightarrow \frac{dz}{du} \left[1 - \left(a \frac{dz}{du} \right)^2 \right] = a \frac{dz}{du} (1 - z)$$



$$1 - a + az = a^2 \left(\frac{dz}{du} \right)^2$$

Taking square root on both sides

$$a \frac{dz}{du} = \sqrt{1 - a + az}$$

$$\frac{a dz}{\sqrt{1 - a + az}} = du$$

Integrating on both sides

$$a \left(\frac{2\sqrt{1 - a + az}}{a} \right) = u + c \quad \because \int \frac{1}{\sqrt{ax}} dx = \frac{1}{a} (2\sqrt{x})$$

$$\boxed{2\sqrt{1 - a + az} = x + ay + c} \quad \because u = x + ay$$

This is the required complete integral.

Other solutions can be obtained as usual.

Type IV:

Equations of the form $f_1(x, p) = f_2(y, q)$ -----(1)

To find Complete Integral:

Let $f_1(x, p) = f_2(y, q) = a$ (say)

$$\therefore f_1(x, p) = a \quad ; \quad f_2(y, q) = a$$

From the above we get $p = f_1(x, a) \quad ; \quad q = f_2(y, a)$

Substitute the value of p & q in $z = \int p dx + \int q dy$

Integrating we get complete integral

Other solutions can obtain as usual.

1. **Solve** $p^2 + q^2 = x^2 + y^2$

Solution:



Given $p^2 + q^2 = x^2 + y^2$

$$p^2 - x^2 = y^2 - q^2 \text{ ----- (1)}$$

This is of the form $f_1(x, p) = f_2(y, q)$

To find Complete Integral:

Let $p^2 - x^2 = y^2 - q^2 = a^2$ (say)

$$\therefore p^2 - x^2 = a^2 ; y^2 - q^2 = a^2$$

$$\therefore p^2 = a^2 + x^2 ; q^2 = y^2 - a^2$$

$$\boxed{p = \sqrt{a^2 + x^2}} ; \boxed{q = \sqrt{y^2 - a^2}}$$

Substitute the value of p & q in

$$z = \int \sqrt{x^2 + a^2} dx + \int \sqrt{y^2 - a^2} dy$$

$$\boxed{z = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \left(\frac{x}{a} \right) + \frac{y}{2} \sqrt{y^2 - a^2} + \frac{a^2}{2} \cosh^{-1} \left(\frac{y}{a} \right) + c}$$

$$\therefore \mathbf{1) \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \left(\frac{x}{a} \right) \quad \mathbf{2) \int \sqrt{y^2 - a^2} dy = \frac{y}{2} \sqrt{y^2 - a^2} + \frac{a^2}{2} \cosh^{-1} \left(\frac{y}{a} \right)}$$

Integrating we get complete integral

Other solutions can obtain as usual.

2. **Find the complete integral of $p^2 y(1 + x^2) = qx^2$**

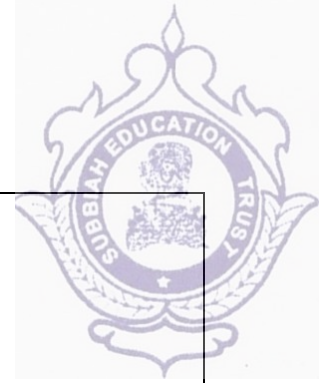
Solution:

Given $p^2 y(1 + x^2) = qx^2$

$$\frac{p^2(1 + x^2)}{x^2} = \frac{q}{y} \text{ ----- (1)}$$

This is of the form $f_1(x, p) = f_2(y, q)$

To find Complete Integral:



$$\text{Let } \frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a \text{ (say)}$$

$$\frac{p^2(1+x^2)}{x^2} = a ; \frac{q}{y} = a$$

$$p^2 = a \frac{x^2}{1+x^2} ; q = ay$$

$$\boxed{p = \frac{\sqrt{a} x}{\sqrt{1+x^2}}} ; \boxed{q = ay}$$

Substitute the value of p & q in

$$z = \int \frac{\sqrt{a} x}{\sqrt{1+x^2}} dx + \int ay dy$$

$$\text{let } 1+x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{dt}{2}$$

$$z = \sqrt{a} \int \frac{1}{\sqrt{t}} \frac{dt}{2} + a \frac{y^2}{2}$$

$$z = \frac{\sqrt{a}}{2} 2\sqrt{t} + \frac{ay^2}{2} \quad \because \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x}$$

$$\boxed{z = \sqrt{a} \sqrt{1+x^2} + \frac{ay^2}{2} + c}$$

3. Find the complete integral of $p + q = \sin x + \sin y$

Solution:

$$\text{Given } p + q = \sin x + \sin y$$

$$p - \sin x = \sin y - q \text{ ----- (1)}$$

This is of the form $f_1(x, p) = f_2(y, q)$

To find Complete Integral:

$$\text{Let } p - \sin x = \sin y - q = a \text{ (say)}$$



$$\therefore p - \sin x = a \quad ; \quad \sin y - q = a$$

$$\therefore \boxed{p = \sin x + a} \quad ; \quad \boxed{q = \sin y - a}$$

Substitute the value of p & q in

$$z = \int (\sin x + a) dx + \int (\sin y - a) dy$$

$$z = \cos x + ax + \cos y - ay + c$$

$$\boxed{z = \cos x + \cos y - a(x - y) + c}$$

This is the required complete integral



DEPARTMENT OF MATHEMATICS

UNIT I - PARTIAL DIFFERENTIAL EQUATIONS

1.3 LAGRANGE'S LINEAR DIFFERENTIAL EQUATION

Lagrange's Linear Differential Equations:

Equations of the form $Pp + Qq = R$ (or) $P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} = R$

Where P, Q, R are functions of x, y, z or constants.

Procedure :

1. Write the auxiliary equation $\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$

2. Solve the auxiliary equation by using

- a) Method of grouping
- b) Method of multipliers

a) Method of grouping: In the auxiliary equation, if the variables can be separated in any pair of equations, then we get a solution of the form $u(x, y, z) = c_1$ & $v(x, y, z) = c_2$

\therefore The general solution is $\boxed{\phi(u, v) = 0}$

b) Method of Multipliers:

i) Choose any three multipliers l, m, n which may be constants or functions of x, y, z we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR}$$

If it is possible to choose l, m, n such that $lP + mQ + nR = 0$ then $l dx + m dy + n dz = 0$



Integrating this we get $u(x, y, z) = c_1$

ii) Choose another any three multipliers l', m', n' which may be constants or functions of x, y, z we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}$$

If it is possible to choose l', m', n' such that $l'P + m'Q + n'R = 0$ then $l'dx + m'dy + n'dz = 0$

Integrating this we get $v(x, y, z) = c_2$

∴ The general solution is $\phi(u, v) = 0$

1. **Solve** $x(y-z)p + y(z-x)q = z(x-y)$

Solution:

Given $x(y-z)p + y(z-x)q = z(x-y)$

This is of the form $Pp + Qq = R$

Where $P = x(y-z)$; $Q = y(z-x)$; $R = z(x-y)$

The auxiliary equation be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \text{ -----(1)}$$

i) Choose the multipliers as (1,1,1)

$$\begin{aligned} (1) \Rightarrow \frac{dx}{x(y-z)} &= \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \\ &= \frac{dx + dy + dz}{xy - xz + yz - xy + xz - yz} = \frac{dx + dy + dz}{0} \end{aligned}$$

$$\therefore dx + dy + dz = 0$$

$$\text{Integrating } \int dx + \int dy + \int dz = 0$$



$$x + y + z = c_1$$

ii) Choose the multipliers as $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

$$(1) \Rightarrow \frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y-z+z-x+x-y} = \frac{dx+dy+dz}{0}$$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating $\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2 \Rightarrow xyz = c_2$$

\therefore The general solution is $\phi(x + y + z, xyz) = 0$

2. Solve $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

Solution:

Given $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

This is of the form $Pp + Qq = R$

Where $P = x(z^2 - y^2)$; $Q = y(x^2 - z^2)$; $R = z(y^2 - x^2)$

The auxiliary equation be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)} \text{ -----(1)}$$



i) Choose the multipliers as (x, y, z)

$$(1) \Rightarrow \frac{xdx}{x^2(z^2 - y^2)} = \frac{ydy}{y^2(x^2 - z^2)} = \frac{zdz}{z^2(y^2 - x^2)}$$

$$= \frac{xdx + ydy + zdz}{x^2z^2 - x^2y^2 + x^2y^2 - y^2z^2 + y^2z^2 - x^2z^2} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0$$

$$\text{Integrating } \int xdx + \int ydy + \int zdz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1^2}{2}$$

$$\boxed{x^2 + y^2 + z^2 = c_1^2}$$

ii) Choose the multipliers as $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

$$(1) \Rightarrow \frac{\frac{dx}{x}}{\frac{x}{x^2(z^2 - y^2)}} = \frac{\frac{dy}{y}}{\frac{y}{y^2(x^2 - z^2)}} = \frac{\frac{dz}{z}}{\frac{z}{z^2(y^2 - x^2)}}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{z^2 - y^2 + x^2 - z^2 + y^2 - x^2} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

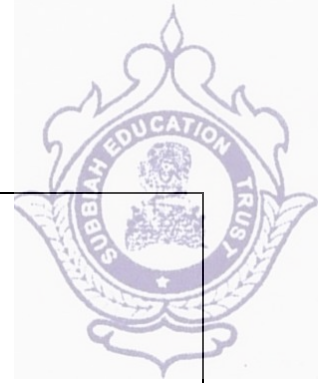
$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\text{Integrating } \int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2 \quad \Rightarrow \quad \boxed{xyz = c_2}$$

$$\therefore \text{The general solution is } \boxed{\phi(x^2 + y^2 + z^2, xyz) = 0}$$



3.

Solve $(x^2 - y^2 - z^2)p + 2xyq = 2zx$

Solution:

Given $(x^2 - y^2 - z^2)p + 2xyq = 2zx$

This is of the form $Pp + Qq = R$

Where $P = (x^2 - y^2 - z^2)$; $Q = 2xy$; $R = 2zx$

The auxiliary equation be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2zx} \text{ -----(1)}$$

i) by method of grouping, from last two ratios

$$(1) \Rightarrow \frac{dy}{2xy} = \frac{dz}{2zx}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\text{Integrating } \int \frac{dy}{y} = \int \frac{dz}{z}$$

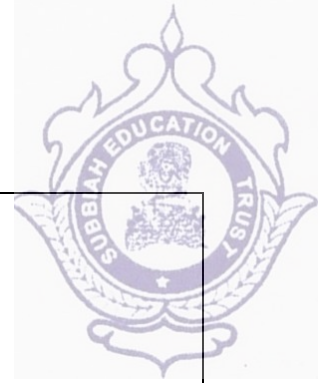
$$\log y = \log z + \log c_1$$

$$\log y - \log z = \log c_1$$

$$\log \frac{y}{z} = \log c_1$$

$$\boxed{\frac{y}{z} = c_1}$$

ii) Choose the multipliers as (x, y, z)



$$\begin{aligned}
 (1) \Rightarrow \frac{xdx}{x(x^2 - y^2 - z^2)} &= \frac{ydy}{y(2xy)} = \frac{zdz}{z(2zx)} \\
 &= \frac{xdx + ydy + zdz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{xdx + ydy + zdz}{x^3 + xy^2 + xz^2} \\
 &= \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}
 \end{aligned}$$

From 3rd and last ratio

$$\frac{dz}{(2zx)} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$$

$$\frac{dz}{z} = \frac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)}$$

$$\text{Integrating } \int \frac{dz}{z} = \int \frac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)}$$

$$\log z = \log(x^2 + y^2 + z^2) \quad \because \int \frac{f'(x)}{f(x)} dx = \log[f(x)]$$

$$\log z = \log(x^2 + y^2 + z^2) + \log c_2$$

$$\log z - \log(x^2 + y^2 + z^2) = \log c_2$$

$$\log \frac{z}{x^2 + y^2 + z^2} = \log c_2$$

$$\boxed{\frac{z}{x^2 + y^2 + z^2} = c_2}$$

$$\text{The general solution is } \boxed{\phi\left(\frac{y}{z}, \frac{z}{x^2 + y^2 + z^2}\right) = 0}$$

4. Solve $(3z - 4y)p + (4x - 2z)q = 2y - 3x$

Hint:



	<p>The multipliers are (x, y, z) & $(2, 3, 4)$</p> <p>The general solution is $\phi(x^2 + y^2 + z^2, 2x + 3y + 4z) = 0$</p>
5.	<p>Solve $(y - xz)p + (yz - x)q = (x + y)(x - y)$</p> <p>Hint:</p> <p>The multipliers are (x, y, z) & $(y, x, 1)$</p> <p>The general solution is $\phi(x^2 + y^2 + z^2, xy + z) = 0$</p>
6.	<p>Solve $x(y^2 + z)p + y(x^2 + z)q = z(x^2 - y^2)$</p> <p>Hint:</p> <p>The multipliers are $\left(\frac{1}{x}, \frac{-1}{y}, \frac{1}{z}\right)$ & $(x, -y, -1)$</p> <p>The general solution is $\phi\left(\frac{xz}{y}, x^2 - y^2 - 2z\right) = 0$</p>



DEPARTMENT OF MATHEMATICS

UNIT I - PARTIAL DIFFERENTIAL EQUATIONS

1.4 HOMOGENEOUS LINEAR PDE OF SECOND AND HIGHER ORDER WITH CONSTANT COEFFICIENTS

Homogeneous Linear PDE of second and higher order with constant co-efficient:

Consider the second order homogeneous linear PDE

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = f(x, y) \text{ ----- (1)}$$

Let the differential operator $D = \frac{\partial}{\partial x}$ & $D' = \frac{\partial}{\partial y}$

$$(1) \Rightarrow (D^2 + a_1 D D' + a_2 D'^2)z = f(x, y) \text{ ----- (2)}$$

The general solution of equation (2) is

$$z = \text{complementary function} + \text{Particular Integral} = \text{C.F.} + \text{P.I.}$$

To find complementary Function:

1. Write the Auxiliary equation by putting $D = m, D' = 1, z = 1$, & $RHS = 0$

$$(2) \Rightarrow m^2 + a_1 m + a_2 = 0$$

2. Solve the auxiliary equation, we get the roots of m . Say the roots are m_1, m_2

3. Comparing the roots of m and write the complementary function.

Case 1: The Roots are real and distinct : say $m_1 \neq m_2$

$$C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$$



Case 2: The Roots are real and equal : say $m_1 = m_2 = m$

$$C.F = f_1(y+mx) + xf_2(y+mx)$$

Note: If the roots are $m = \alpha \pm i\beta$

$$\text{then } C.F = f_1[y+(\alpha+i\beta)] + f_2[y+(\alpha-i\beta)]$$

To find Particular Integral :

Type : I

If $RHS = e^{ax+by}$ then

$$P.I = \frac{1}{f(D, D')} e^{ax+by}$$

$$\text{Rule: } D = a \text{ \& } D' = b$$

$$P.I = \frac{1}{f(a, b)} e^{ax+by}, \text{ Provided Denominator } \neq 0$$

If Denominator = 0, then 1) multiply the numerator by x 2) differentiating denominator partially w.r.to D

$$P.I = \frac{x}{f'(D, D')} e^{ax+by}$$

$$P.I = \frac{1}{f'(a, b)} e^{ax+by}, \text{ Provided Denominator } \neq 0 \therefore \text{ Replace } D = a \text{ \& } D' = b$$

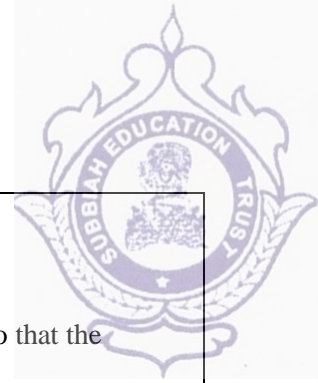
Continuing this process until we get $Dr \neq 0$.

Type : II

If $RHS = \sin(ax+by)$ (or) $RHS = \cos(ax+by)$ then

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by)$$

$$\text{Rule: } D^2 = -(a^2); DD' = (-ab); D'^2 = -(b^2)$$



$$P.I = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by) \text{ , Provided } D \neq 0$$

Note 1: After substitutions the denominator will be in terms of D & D' . Multiply and divide by D so that the denominator will have D^2 & DD' terms.

Note 2: After substitutions the denominator will be in terms of D & constant terms,

For eg.
$$P.I = \frac{1}{D-5} \sin(x-2y)$$

Take conjugate of denominator with constant term and multiplied with both numerator and denominator.

$$P.I = \frac{1}{D-5} \times \frac{D+5}{D+5} \sin(x-2y) = \frac{1}{D^2-25} \sin(x-2y)$$

Then apply the rule as usual.

Type : III

If $RHS = x^m y^n$ (polynomial type) then

$$P.I = \frac{1}{f(D, D')} x^m y^n \text{ , we bring this into a standard binomial format, by taking out highest power term of } D.$$

$$(i.e) P.I = [1 \pm f(D, D')]^{-1} x^m y^n$$

This will be expanded by using the formulae

$$(1-x)^{-1} = 1+x+x^2+x^3+\dots$$

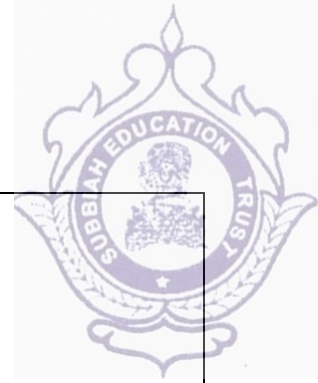
$$(1+x)^{-1} = 1-x+x^2-x^3+\dots$$

$$(1-x)^{-2} = 1+2x+3x^2+4x^3+\dots$$

$$(1+x)^{-2} = 1-2x+3x^2-4x^3+\dots$$

Note:

$$\frac{1}{D} = \int dx \text{ , } D' = \frac{\partial}{\partial y}$$

**Type : IV (Exponential shifted rule)**

If $\boxed{RHS = e^{ax+by} \cos(ax+by)}$ (or) $\boxed{RHS = e^{ax+by} \sin(ax+by)}$ (or) $\boxed{RHS = e^{ax+by} x^m y^n}$ then

$$P.I = \frac{1}{f(D, D')} e^{ax+by} \sin(ax+by)$$

$$P.I = e^{ax+by} \frac{1}{f(D+a, D'+b)} \sin(ax+by)$$

Here after apply the rule as we discussed in Type II&III

Type : V

If $\boxed{RHS = y \cos x}$ (or) $\boxed{RHS = y \sin x}$ then

Case 1:

$$P.I = \frac{1}{D - mD'} y \cos x$$

$$P.I = \frac{1}{D - mD'} \int (c - mx) \cos x dx \quad \because \text{Rule: } \boxed{y = c - mx}$$

Case 2:

$$P.I = \frac{1}{D + mD'} y \cos x$$

$$P.I = \frac{1}{D + mD'} \int (c + mx) \cos x dx \quad \because \text{Rule: } y = c - mx$$

Note: After integration we have to replace $c - mx = y$

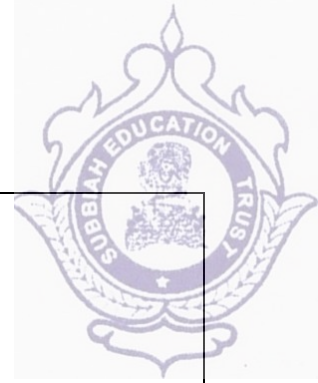
1. **Solve** $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x - y)$

Solution:

Given $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x - y)$

To find C.F

The auxiliary equation is



$$m^2 - m - 20 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m - 5)(m + 4) = 0$$

$$\boxed{m = 5, -4}$$

$$\therefore \boxed{C.F = f_1(y - 4x) + f_2(y + 5x)} \quad \because \text{The roots are real and distinct} \Rightarrow C.F = f_1(y + mx) + x f_2(y + mx)$$

To find P.I

$$P.I = \frac{1}{D^2 - DD' - 20D'^2} [e^{5x+y} + \sin(4x - y)]$$

$$P.I = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y} \quad \text{Rule: replace } D = 5 \text{ \& } D' = 1 \quad \text{Type:1}$$

$$= \frac{1}{25 - 5 - 20} e^{5x+y} = \frac{1}{0} e^{5x+y} \quad \because \text{Introducing } x \text{ in Nr.Diff Dr. partially w.r.to } D$$

$$= \frac{x}{2D - D'} e^{5x+y} \quad \text{Rule: replace } D = 5 \text{ \& } D' = 1$$

$$= \frac{x}{10 - 1} e^{5x+y}$$

$$\boxed{P.I_1 = \frac{x}{9} e^{5x+y}}$$

$$P.I_2 = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x - y) \quad \text{here } a = 4 \text{ \& } b = -1 \quad (\text{Type:2})$$

$$\text{Rule: replace } D^2 = -(a^2) = -16; D'^2 = -(b^2) = -1 \text{ \& } DD' = -(ab) = 4$$

$$P.I_2 = \frac{1}{-16 - 4 - 20(-1)} \sin(4x - y) = \frac{1}{0} \sin(4x - y)$$

$$= \frac{x}{2D - D'} \sin(4x - y) \quad \because \text{Introducing } x \text{ in Nr.Diff Dr. partially w.r.to } D$$

$$= \frac{x}{2D - D'} \times \frac{D}{D} \sin(4x - y)$$



$$= \frac{x D}{2D^2 - DD'} \sin(4x - y)$$

$$= \frac{x D}{2(-16) - 4} \sin(4x - y)$$

$$= \frac{x D (\sin(4x - y))}{-32 - 4}$$

$$P.I_2 = \frac{4x \cos(4x + 3)}{-36} = \frac{-1}{9} x \cos(4x + 3) \quad \because \frac{d}{dx} (\sin nx) = n \cos nx$$

$$P.I_2 = \frac{-1}{9} x \cos(4x + 3)$$

$$P.I = \frac{x}{9} e^{5x+y} - \frac{1}{9} x \cos(4x + 3) = \frac{x}{9} [e^{5x+y} - \cos(4x + 3)]$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y - 4x) + f_2(y + 5x) + \frac{x}{9} [e^{5x+y} - \cos(4x + 3)]$$

2. Solve $\frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^3 z}{\partial x^2 \partial x} = e^{x+2y} + 4 \sin(x + y)$

Solution: same as previous problem

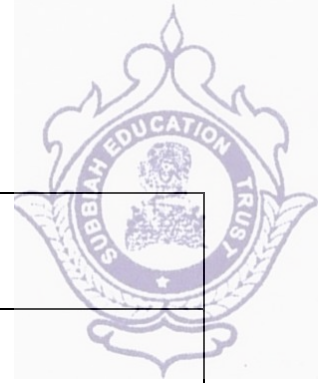
Hint:

Given $\frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^3 z}{\partial x^2 \partial x} = e^{x+2y} + 4 \sin(x + y)$

$$(D^3 - 2D^2 D') z = e^{x+2y} + 4 \sin(x + y) \quad \because D = \frac{\partial}{\partial x} \quad \& \quad D' = \frac{\partial}{\partial y}$$

$$m = 0, 0, 2$$

$$C.F = f_1(y) + x f_2(y) + f_2(y + 2x)$$



$$P.I = \frac{-1}{3} e^{x+2y} - 4 \cos(x+2y)$$

3. Solve $(D^2 + DD' - 6D'^2)z = x^2y + e^{3x+y}$

Solution:

Given $(D^2 + DD' - 6D'^2)z = x^2y + e^{3x+y}$

To find C.F

The auxiliary equation is

$$m^2 + m - 6 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m-2)(m+3) = 0$$

$$m = 2, -3$$

$$\therefore C.F = f_1(y-3x) + f_2(y+2x) \quad \because \text{The roots are real and distinct} \Rightarrow C.F = f_1(y+m_1x) + f_2(y+m_2x)$$

To find P.I

$$P.I = \frac{1}{D^2 + DD' - 6D'^2} [x^2y + e^{3x+y}]$$

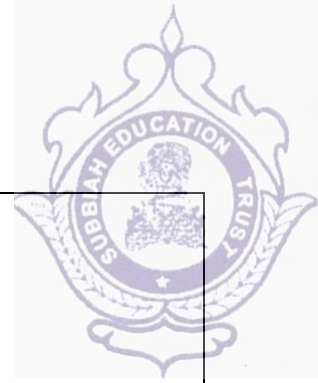
$$P.I = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2 + DD' - 6D'^2} x^2y \quad (\text{Type:3})$$

$$= \frac{1}{D^2 \left[1 + \left(\frac{DD' - 6D'^2}{D^2} \right) \right]} x^2y$$

$$= \frac{1}{D^2} \left[1 + \left(\frac{D'}{D} - \frac{6D'^2}{D^2} \right) \right]^{-1} x^2y$$

$$= \frac{1}{D^2} \left[1 - \left(\frac{D'}{D} - \frac{6D'^2}{D^2} \right) + \dots \right] x^2y \quad \because (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$



$$= \frac{1}{D^2} \left[1 - \frac{D'}{D} + \frac{6D'^2}{D^2} \right] x^2 y \quad \because D' = \frac{\partial}{\partial y}$$

$$= \frac{1}{D^2} \left[x^2 y - \frac{D'(x^2 y)}{D} + \frac{6D'^2(x^2 y)}{D^2} \right]$$

$$= \frac{1}{D^2} \left[x^2 y - \frac{x^2}{D} \right]$$

$$= \frac{1}{D^2} \left[x^2 y - \int x^2 dx \right] \quad \because \frac{1}{D} = \int dx$$

$$= \frac{1}{D^2} \left[x^2 y - \frac{x^3}{3} \right]$$

$$= \frac{1}{D} \int \left(x^2 y + \frac{x^3}{3} \right) dx$$

$$= \int \left(\frac{x^3 y}{3} - \frac{x^4}{12} \right) dx$$

$$P.I_1 = \frac{x^4 y}{12} - \frac{x^5}{60}$$

$$P.I_2 = \frac{1}{D^2 + DD' - 6D'^2} e^{3x+y} \quad \text{Rule: replace } D = a = 3 \text{ \& } D' = b = 1 \quad \text{Type: 1}$$

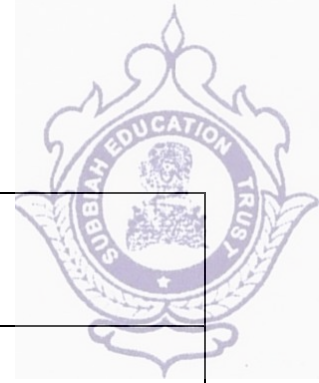
$$= \frac{1}{9 + 3 - 6(1)} e^{3x+y} = \frac{1}{6} e^{3x+y}$$

$$P.I_2 = \frac{1}{6} e^{3x+y}$$

$$P.I = \frac{x^4 y}{12} - \frac{x^5}{60} + \frac{1}{6} e^{3x+y}$$

The general solution is

$$z = C.F + P.I$$



$$z = f_1(y-3x) + f_2(y+2x) + \frac{x^4 y}{12} - \frac{x^5}{60} + \frac{1}{6} e^{3x+y}$$

4. Solve $(D^2 + 2DD' + D'^2)z = x^2 y + e^{x-y}$

Solution: same as previous problem

Hint:

$$m = -1, -1$$

$$C.F = f_1(y-x) + x f_2(y-x)$$

$$P.I = \frac{x^4 y}{12} - \frac{x^5}{30} + \frac{x^2}{2} e^{x-y}$$

5. Solve $(D^2 - 6DD' + 5D'^2)z = xy + e^x \sinh y$

Solution:

$$\text{Given } (D^2 - 6DD' + 5D'^2)z = xy + e^x \sinh y$$

$$= xy + e^x \left(\frac{e^y - e^{-y}}{2} \right) \quad \because \sinh \theta = \frac{e^\theta - e^{-\theta}}{2}, \text{ here } \theta = y$$

$$= xy + \left(\frac{e^x e^y - e^x e^{-y}}{2} \right)$$

$$(D^2 - 6DD' + 5D'^2)z = xy + \frac{e^{x+y}}{2} - \frac{e^{x-y}}{2} \quad \because e^a e^b = e^{a+b}$$

To find C.F

The auxiliary equation is

$$m^2 - 6m + 5 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m-1)(m-5) = 0$$

$$m = 1, 2$$

$$\therefore C.F = f_1(y+x) + f_2(y+5x) \quad \because \text{The roots are real and distinct } \Rightarrow C.F = f_1(y+m_1x) + x f_2(y+m_2x)$$



To find P.I

$$P.I = \frac{1}{D^2 - 6DD' + 5D'^2} \left[xy + \frac{e^{x+y}}{2} - \frac{e^{x-y}}{2} \right]$$

$$P.I = P.I_1 + P.I_2 - P.I_3 \quad \text{-----(1)}$$

To find P.I₁

$$P.I_1 = \frac{1}{D^2 - 6DD' + 5D'^2} xy \quad \text{(Type:3)}$$

$$= \frac{1}{D^2 \left[1 + \left(\frac{-6DD' + 5D'^2}{D^2} \right) \right]} xy$$

$$= \frac{1}{D^2} \left[1 + \left(\frac{-6D'}{D} + \frac{5D'^2}{D^2} \right) \right]^{-1} xy$$

$$= \frac{1}{D^2} \left[1 - \left(\frac{-6D'}{D} + \frac{5D'^2}{D^2} \right) + \dots \right] xy \quad \because (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$= \frac{1}{D^2} \left[1 + \frac{6D'}{D} - \frac{5D'^2}{D^2} + \dots \right] xy$$

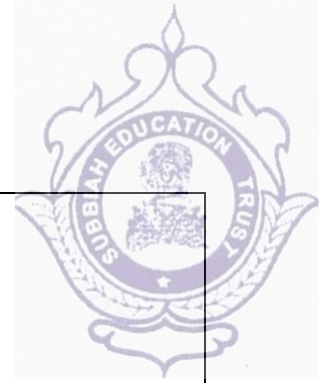
$$= \frac{1}{D^2} \left[xy + \frac{6x}{D} - \frac{5(0)}{D^2} + \dots \right] xy \quad \because D'(y) = \frac{\partial}{\partial y}(y) = 1 \quad \& \quad D'^2(y) = \frac{\partial^2}{\partial y^2}(y) = 0$$

$$= \frac{1}{D^2} \left[xy + 6 \int x dx \right] = \frac{1}{D^2} \left[xy + \frac{6x^2}{2} \right] \quad \because \frac{1}{D} = \int dx$$

$$= \frac{1}{D} \int (xy + 3x^2) dx = \frac{1}{D} \left(\frac{x^2 y}{2} + \frac{3x^3}{3} \right) = \int \left(\frac{x^2 y}{2} + x^3 \right) dx$$

$$P.I_1 = \frac{x^3 y}{6} + \frac{x^4}{4}$$

To find P.I₂



$$P.I_2 = \frac{1}{D^2 - 6DD' + 5D'^2} \frac{e^{x+y}}{2} \quad \text{Type:1}$$

$$= \left(\frac{1}{2}\right) \frac{1}{1-6+5} e^{x+y} = \left(\frac{1}{2}\right) \frac{1}{0} e^{x+y} \quad \text{Rule: replace } D = a = 1 \text{ \& } D' = b = 1$$

$$= \left(\frac{1}{2}\right) \frac{x}{2D - 6D' + 0} e^{x+y}$$

$$= \left(\frac{1}{2}\right) \frac{x}{2-6} e^{x+y}$$

$$= \left(\frac{1}{2}\right) \frac{-x}{4} e^{x+y}$$

$$P.I_2 = \frac{-x}{8} e^{x+y}$$

To find P.I₃

$$P.I_3 = \frac{1}{D^2 - 6DD' + 5D'^2} \frac{e^{x-y}}{2} \quad \text{Type:1}$$

$$= \left(\frac{1}{2}\right) \frac{1}{1+6+5} e^{x-y} \quad \text{Rule: replace } D = a = 1 \text{ \& } D' = b = -1$$

$$P.I_3 = \frac{1}{24} e^{x-y}$$

$$(1) \Rightarrow P.I = P.I_1 + P.I_2 - P.I_3 = \frac{x^3 y}{6} + \frac{x^4}{4} - \frac{x}{8} e^{x+y} - \frac{1}{24} e^{x+y}$$

The general solution is

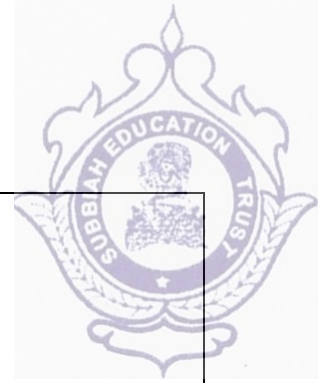
$$z = C.F + P.I$$

$$z = f_1(y+x) + f_2(y+5x) + \frac{x^3 y}{6} + \frac{x^4}{4} - \frac{x}{8} e^{x+y} - \frac{1}{24} e^{x+y}$$

6. Solve $(D^2 + 2DD' + D'^2)z = \sinh(x+y) + e^{x+2y}$



	<p>Solution: same as previous problem</p> <p>Hint:</p> $(D^2 + 2DD' + D'^2)z = \sinh(x+y) + e^{x+2y}$ $(D^2 + 2DD' + D'^2)z = \frac{e^{x+y} - e^{-(x-y)}}{2} + e^{x+2y} = \frac{e^{x+y}}{2} - \frac{e^{-x-y}}{2} + e^{x+2y}$ $(D^2 + 2DD' + D'^2)z = \frac{e^{x+y}}{2} - \frac{e^{-x-y}}{2} + e^{x+2y}$ <p>$m = -1, -1$</p> <p>$C.F = f_1(y-x) + xf_2(y-x)$</p> $P.I = \frac{e^{x+y}}{8} - \frac{e^{-x-y}}{8} - \frac{e^{x+2y}}{9}$
7.	<p>Solve $(D^2 - 4DD' + 4D'^2)z = e^{x+2y}$</p> <p>Solution:</p> <p>Hint: $m = 2, 2$</p> <p>$C.F = f_1(y+x) + f_2(y+5x)$</p> $P.I = \frac{x^2}{2} e^{2x+y}$
8.	<p>Solve $(2D^2 - 5DD' + 2D'^2)z = 5 \sin(2x+y)$</p> <p>Solution:</p> <p>Given $(2D^2 - 5DD' + 2D'^2)z = 5 \sin(2x+y)$</p> <p>To find C.F</p> <p>The auxiliary equation is</p> $2m^2 - 5m + 2 = 0 \quad \therefore \text{replace } D = m, D' = 1, z = 0$



$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{here } a = 2, b = -5, c = 2$$

$$= \frac{-(-5) \pm \sqrt{25 - 4(2)(2)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm \sqrt{9}}{4}$$

$$= \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm \sqrt{9}}{4} = \frac{5 \pm 3}{4}$$

$$m = \frac{5+3}{4}, m = \frac{5-3}{4} \Rightarrow \boxed{m = 2, m = \frac{1}{2}}$$

$$\therefore \boxed{C.F = f_1(y + 2x) + f_2\left(y + \frac{x}{2}\right)} \quad \because \text{The roots are real and distinct} \Rightarrow C.F = f_1(y + m_1x) + f_2(y + m_2x)$$

To find P.I

$$P.I = \frac{1}{2D^2 - 5DD' + 2D'^2} 5 \sin(2x + y) \quad \text{Type:2}$$

$$P.I = \frac{1}{2D^2 - 5DD' + 2D'^2} 5 \sin(2x + y) \quad \text{here } a = 2 \& b = 1$$

Rule: replace $D^2 = -(a^2) = -4$; $D'^2 = -(b^2) = -1$ & $DD' = -(ab) = -2$

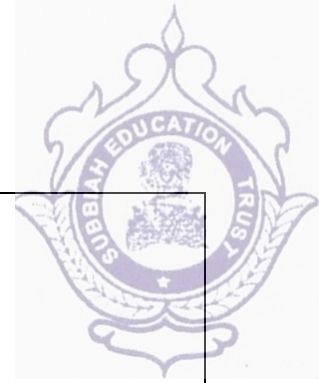
$$P.I = \frac{1}{2(-4) - 5(-2) + 2(-1)} 5 \sin(2x + y)$$

$$P.I = \frac{1}{-8 + 10 - 2} 5 \sin(2x + y) = \frac{1}{0} 5 \sin(2x + y)$$

$$= \frac{x}{4D - 5D' + 0} 5 \sin(2x + y) \quad \{\text{Introducing } x \text{ in Nr. \& Diff } D \text{ partially w.r. to } D\}$$

$$= \frac{x}{4D - 5D'} \times \frac{D}{D} 5 \sin(2x + y)$$

$$= 5 \frac{x D}{4D^2 - 5DD'} \sin(2x + y)$$



$$= 5 \frac{x D}{4(-4) - 5(-2)} \sin(2x + y) = 5 \frac{x D [\sin(2x + y)]}{-16 + 10}$$

$$= \frac{5}{-6} x [2 \cos(2x + y)]$$

$$P.I = \frac{-5}{3} x \cos(2x + y)$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y + 2x) + f_2\left(y + \frac{x}{2}\right) - \frac{5}{3} x \cos(2x + y)$$

9. Solve the equation $(D^3 + D^2 D' - 4 D D'^2 - 4 D'^3) z = \cos(2x + y)$

Solution:

$$\text{Given } (D^3 + D^2 D' - 4 D D'^2 - 4 D'^3) z = \cos(2x + y)$$

To find C.F

The auxiliary equation is

$$m^3 + m^2 - 4m - 4 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$m^2(m + 1) - 4(m + 1) = 0$$

$$(m + 1)(m^2 - 4) = 0$$

$$m = -1, -2, 2$$

$$\therefore C.F = f_1(y - x) + f_2(y - 2x) + f_3(y + 2x)$$

To find P.I

$$P.I = \frac{1}{D^3 + D^2 D' - 4 D D'^2 - 4 D'^3} \cos(2x + y) \quad \text{here } a = 2, b = 1 \quad (\text{Type:2})$$

Rule: replace $D^2 = -(a^2) = -4$; $D'^2 = -(b^2) = -1$ & $DD' = -(ab) = -2$



$$P.I = \frac{1}{-4D - 4D' - 4D(-1) - 4(-1)D'} \cos(2x + y)$$

$$= \frac{1}{-4D - 4D' + 4D + 4D'} \cos(2x + y) = \frac{1}{0} \cos(2x + y)$$

$$P.I = \frac{x}{3D^2 + 2DD' - 4D'^2 - 0} \cos(2x + y) \quad \{\text{Introducing } x \text{ in Nr. \& Diff Dr. partially w.r.to } D\}$$

$$= \frac{x}{3(-4) + 2(-2) - 4(-1)} \cos(2x + y) = \frac{x}{-12 - 4 + 4} \cos(2x + y)$$

$$P.I = \frac{-x}{12} \cos(2x + y)$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y - x) + f_2(y - 2x) + f_3(y + 2x) - \frac{x}{12} \cos(2x + y)$$

10. Solve $(D^3 - 7DD'^2 - 6D'^3)z = \cos(x + 2y) + 4$

Solution:

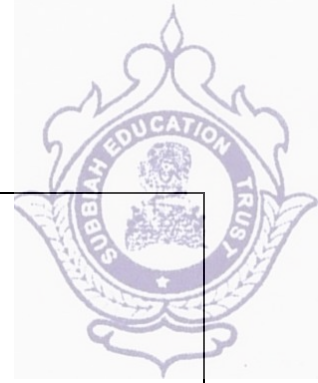
Given $(D^3 - 7DD'^2 - 6D'^3)z = \cos(x + 2y) + 4$

To find C.F

The auxiliary equation is

$$m^3 - 7m - 6 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

m=-1	1	0	-7	6
	0	-1	1	6
	1	-1	-6	0



$$m = -1, m^2 - m - 6 = 0$$

$$m = -1, (m-3)(m+2) = 0$$

$$m = -1, -2, 3$$

$$\therefore C.F = f_1(y-x) + f_2(y-2x) + f_3(y+3x)$$

To find P.I

$$P.I = \frac{1}{D^3 - 7DD'^2 - 6D'^3} [\cos(x+2y) + 4]$$

$$P.I = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \cos(x+2y) \quad \text{here } a=1, b=2 \quad (\text{Type:2})$$

Rule: replace $D^2 = -(a^2) = -1$; $D'^2 = -(b^2) = -4$ & $DD' = -(ab) = -2$

$$P.I_1 = \frac{1}{(-1)D - 7D(-2) - 6(-2)D'} \cos(x+2y) = \frac{1}{-D + 14D + 12D'} \cos(x+2y)$$

$$= \frac{1}{13D + 12D'} \times \frac{D}{D} \cos(x+2y)$$

$$= \frac{D}{13D^2 + 12D'D} \cos(x+2y)$$

$$= \frac{D}{13(-1) + 12(-2)} \cos(x+2y) = \frac{D \cos(x+2y)}{-13 - 24}$$

$$P.I_1 = \frac{-\sin(x+2y)}{37}$$

To find P.I₂



$$\begin{aligned}
 P.I_2 &= \frac{1}{D^3 - 7DD'^2 - 6D'^3} 4e^{0x+0y} \quad \because e^0 = 1 \quad \text{Type:1} \\
 &= \frac{1}{D^3 - 7DD'^2 - 6D'^3} 4e^{0x+0y} = \frac{1}{0} 4e^{0x+0y} \quad \text{Rule: Replace } D = 0, D' = 0 \\
 &= \frac{x}{3D^2 - 7D'^2 - 0} 4e^{0x+0y} = \frac{x}{0} 4e^{0x+0y} \\
 &= \frac{x}{6D - 0} 4e^{0x+0y} = \frac{x}{0} 4e^{0x+0y} \\
 &= \frac{x^2}{6} 4
 \end{aligned}$$

$$P.I_2 = \frac{2x^2}{3}$$

$$P.I = \frac{-1}{37} \sin(x+2y) + \frac{2x^2}{3}$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{37} \sin(x+2y) + \frac{2x^2}{3}$$

11. Solve $(D^2 + 3DD' - 4D'^2)z = xy + \cos(2x + y)$

Solution: same as previous problem

Hint:

$$m = -4, 1$$

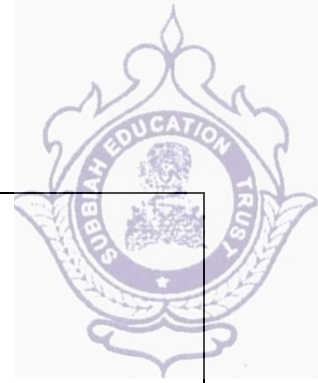
$$C.F = f_1(y - 4x) + xf_2(y + x)$$

$$P.I = \frac{-1}{6} \cos(2x + y) + \frac{x^3 y}{6} - \frac{x^4}{8}$$

12. Solve $(D^2 + 3DD' - 4D'^2)z = x + \sin y$



	<p>Hint:</p> $m = -4, 1$ $C.F = f_1(y - 4x) + xf_2(y + x)$ $P.I = \frac{1}{D^2 + 3DD' - 4D'^2} [x + \sin(0x + y)] = \dots = \frac{x^3}{6} + \frac{\sin y}{4}$
<p>13.</p>	<p>Solve $(D^2 - DD' - 2D'^2)z = (2x + 3y) + e^{3x+4y}$</p> <p>Hint:</p> $m = -1, 2$ $C.F = f_1(y - x) + f_2(y + 2x)$ $P.I = \frac{5x^3}{6} + \frac{3x^2y}{2} + \frac{1}{35}e^{3x+4y}$
<p>14.</p>	<p>Solve $(D^2 - 2DD' + D'^2)z = (2 + 4x)e^{x+2y}$</p> <p>Solution:</p> <p>Given $(D^2 - 2DD' + D'^2)z = x^2y^2e^{x+2y}$</p> <p>To find C.F</p> <p>The auxiliary equation is</p> $m^2 - 2m + 1 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$ $(m - 1)(m - 1) = 0$ $\boxed{m = 1, 1}$ <p>$\therefore \boxed{C.F = f_1(y + x) + xf_2(y + x)}$</p> <p>To find P.I</p> $P.I = \frac{1}{D^2 - 2DD' + D'^2} (2 + 4x)e^{x+2y} \quad \text{here } a = 2, b = 1 \quad (\text{Type:4})$



$$P.I = \frac{1}{(D-D')^2} (2+4x) e^{x+2y}$$

Rule: replace $D = D + a = D + 1$; $D' = D' + b = D' + 2$

$$P.I = e^{x+2y} \frac{1}{(D+1-D'-2)^2} (2+4x)$$

$$= e^{x+2y} \frac{1}{(D-D'-1)^2} (2+4x)$$

$$= e^{x+2y} \frac{1}{[-(1-D+D')]^2} (2+4x) = e^{x+2y} \frac{1}{[1-(D-D')]^2} (2+4x)$$

$$= e^{x+2y} [1-(D-D')]^{-2} (2+4x)$$

$$= e^{x+2y} [1+2(D-D')+3(D-D')^2 + \dots] (2+4x) \quad \because (1-x)^{-2} = 1+2x+3x^2+4x^3 + \dots$$

$$= e^{x+2y} [1+2D-2D'+3(D^2-2DD'+D'^2) + \dots] (2+4x)$$

$$= e^{x+2y} [1+2D-2D'+3D^2-6DD'+3D'^2] (2+4x)$$

$$= e^{x+2y} [1+2D+3D^2] (2+4x) \quad \because \text{there is no } y \text{ term in RHS, neglect the term } D'$$

$$= e^{x+2y} [(2+4x) + 2D(2+4x) + 3D^2(2+4x)]$$

$$= e^{x+2y} [2+4x+2(4)+0]$$

$$P.I = e^{x+2y} [4x+10]$$

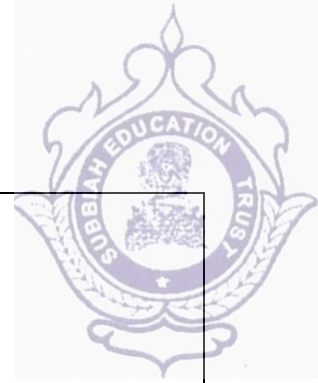
The general solution is

$$z = C.F + P.I$$

$$z = f_1(y+x) + x f_2(y+x) + e^{x+2y} [4x+10]$$

15.

Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x-y} \sin(2x+3y)$

**Solution:**

$$\text{Given } \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x-y} \sin(2x+3y)$$

$$(D^2 - D'^2)z = e^{x-y} \sin(2x+3y)$$

To find C.F

The auxiliary equation is

$$m^2 - 1 = 0 \quad \therefore \text{replace } D = m, D' = 1, z = 0$$

$$m^2 = 1 \Rightarrow \boxed{m = -1, 1}$$

$$\therefore \boxed{C.F = f_1(y-x) + f_2(y+x)}$$

To find P.I

$$P.I = \frac{1}{D^2 - D'^2} e^{x-y} \sin(2x+3y) \quad \text{here } a = 1, b = -1 \quad (\text{Type:4})$$

Rule: replace $D = D + a = D + 1$; $D' = D' + b = D' - 1$

$$P.I = e^{x-y} \frac{1}{(D+1)^2 - (D'-1)^2} \sin(2x+3y)$$

$$= e^{x-y} \frac{1}{D^2 + 2D + 1 - D'^2 + 2D' - 1} \sin(2x+3y)$$

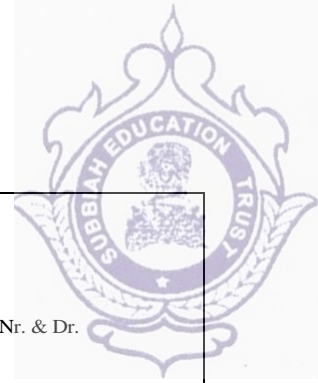
$$= e^{x-y} \frac{1}{D^2 + 2D - D'^2 + 2D'} \sin(2x+3y) \quad \text{Here } a = 2, b = 3$$

Rule: replace $D^2 = -(a^2) = -4$; $D'^2 = -(b^2) = -9$ & $DD' = -(ab) = -6$

$$= e^{x-y} \frac{1}{-4 + 2D - (-9) + 2D'} \sin(2x+3y) = e^{x-y} \frac{1}{2D + 2D' + 5} \sin(2x+3y)$$

$$= e^{x-y} \frac{1}{2D + 2D' + 5} \times \frac{D}{D} \sin(2x+3y)$$

$$= e^{x-y} \frac{D}{2D^2 + 2DD' + 5D} \sin(2x+3y)$$



$$= e^{x-y} \frac{D}{-8-12+5D} \sin(2x+3y) = e^{x-y} \frac{D}{5D-20} \sin(2x+3y)$$

If we multiply and divide by D, we can not get the term D^2, D'^2 term, so we take conjugate for constant term and multiplied with both Nr. & Dr.

$$= e^{x-y} \frac{D}{5D-20} \times \frac{5D+20}{5D+20} \sin(2x+3y)$$

$$= e^{x-y} \frac{5D^2+20D}{25D^2-400} \sin(2x+3y)$$

$$= e^{x-y} \frac{5D^2 \sin(2x+3y) + 20D \sin(2x+3y)}{25(-4) - 400}$$

$$= e^{x-y} \frac{5D \cos(2x+3y) \times 2 + 20 \cos(2x+3y) \times 2}{-100 - 400}$$

$$= e^{x-y} \frac{-20 \sin(2x+3y) + 40 \cos(2x+3y)}{-500}$$

$$P.I = \frac{e^{x-y}}{25} [\sin(2x+3y) - 2 \cos(2x+3y)]$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y-x) + f_2(y+x) + \frac{e^{x-y}}{25} [\sin(2x+3y) - 2 \cos(2x+3y)]$$

16. Solve $(D^2 + DD' - 6D'^2)z = y \cos x$

Solution:

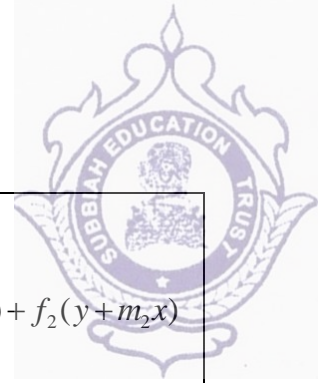
$$\text{Given } (D^2 + DD' - 6D'^2)z = y \cos x$$

To find C.F

The auxiliary equation is

$$m^2 + m - 6 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m-2)(m+3) = 0$$



$$m = 2, -3$$

$$\therefore C.F = f_1(y - 3x) + f_2(y + 2x) \quad \because \text{The roots are real and distinct} \Rightarrow C.F = f_1(y + m_1x) + f_2(y + m_2x)$$

To find P.I

$$P.I = \frac{1}{D^2 + DD' - 6D'^2} y \cos x \quad \text{Type: 5}$$

$$= \frac{1}{(D + 3D')(D - 2D')} y \cos x$$

$$= \frac{1}{(D + 3D')} \int (c - 2x) \cos x \, dx \quad \because \text{Rule: } y = c - mx \text{ here } m = 2$$

$$= \frac{1}{(D + 3D')} [(c - 2x)(\sin x) - (-2)(-\cos x)] \quad \because \int u v dx = uv_1 - u'v_2 + \dots$$

$$= \frac{1}{(D + 3D')} [y \sin x - 2 \cos x]$$

$$= \frac{1}{(D + 3D')} \int [(c + 3x) \sin x - 2 \cos x] dx \quad \because \text{Rule: } y = c + mx \text{ here } m = 3$$

$$= (c + 3x)(-\cos x) - (3)(-\sin x) - 2 \sin x \quad \because y = c + 3x$$

$$= -y \cos x + 3 \sin x - 2 \sin x$$

$$P.I = \sin x - y \cos x$$

The general solution is

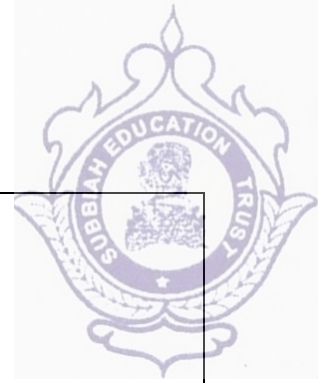
$$z = C.F + P.I$$

$$z = f_1(y - 3x) + f_2(y + 2x) + \sin x - y \cos x$$

17. Solve $(D^2 - 5DD' + 6D'^2)z = y \sin x$

Solution:

Given $(D^2 - 5DD' + 6D'^2)z = y \sin x$

**To find C.F**

The auxiliary equation is

$$m^2 - 5m + 6 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m - 2)(m - 3) = 0$$

$$m = 2, 3$$

$$\therefore C.F = f_1(y + 3x) + f_2(y + 2x) \quad \because \text{The roots are real and distinct} \Rightarrow C.F = f_1(y + m_1x) + f_2(y + m_2x)$$

To find P.I

$$P.I = \frac{1}{D^2 - 5DD' + 6D'^2} y \sin x \quad \text{Type: 5}$$

$$= \frac{1}{(D - 3D')(D - 2D')} y \sin x$$

$$= \frac{1}{(D - 3D')} \int (c - 2x) \sin x \, dx \quad \because \text{Rule: } y = c - mx \text{ here } m = 2$$

$$= \frac{1}{(D - 3D')} [(c - 2x)(-\cos x) - (-2)(-\sin x)] \quad \because \int uv \, dx = uv_1 - u'v_2 + \dots$$

$$= \frac{1}{(D - 3D')} [-y \cos x - 2 \sin x] = \frac{-1}{(D - 3D')} [y \cos x + 2 \sin x]$$

$$= -\int [(c - 3x) \cos x + 2 \sin x] \, dx \quad \because \text{Rule: } y = c - mx \text{ here } m = 3$$

$$= -[(c - 3x)(\sin x) - (-3)(-\cos x) + 2(-\cos x)] \quad \because y = c - 3x$$

$$= -[y \sin x - 3 \cos x - 2 \cos x]$$

$$P.I = 5 \cos x - y \sin x$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y + 3x) + f_2(y + 2x) + 5 \cos x - y \sin x$$



DEPARTMENT OF MATHEMATICS

UNIT I - PARTIAL DIFFERENTIAL EQUATIONS

1.5 NON HOMOGENEOUS LINEAR PDE OF SECOND AND HIGHER ORDER WITH CONSTANT COEFFICIENTS

Non-Homogeneous Linear PDE of second and higher order with constant co-efficient:

Consider the second order non-homogeneous linear PDE

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} + a_3 \frac{\partial z}{\partial x} + a_4 \frac{\partial z}{\partial y} = f(x, y) \text{ ----- (1)}$$

Let the differential operator $D = \frac{\partial}{\partial x}$ & $D' = \frac{\partial}{\partial y}$

$$(1) \Rightarrow (D^2 + a_1 DD' + a_2 D'^2 + a_3 D + a_4 D')z = f(x, y) \text{ ----- (2)}$$

The general solution of equation (2) is

$$z = \text{complementary function} + \text{Particular Integral} = \text{C.F} + \text{P.I}$$

To find complementary Function:

Case : I

The given PDE will bring into the form of $(D - m_1 D' - C_1)(D - m_2 D' - C_2)z = 0$

$$C.F = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x)$$

Case : II

The given PDE will bring into the form of $(D - mD' - C)^2 z = 0$

$$C.F = e^{cx} f_1(y + mx) + xe^{cx} f_2(y + mx)$$

Note: Particular Integral can be obtained, similar like in Homogeneous types.

1. Solve $(D^2 + 2DD' + D'^2 - 2D - 2D')z = e^{3x+y} + \sin(x + 2y)$

Solution:

Given $(D^2 + 2DD' + D'^2 - 2D - 2D')z = e^{3x+y} + \sin(x + 2y)$

To find C.F

$$((D + D')^2 - 2(D + D'))z = 0$$

$$(D + D')(D + D' - 2)z = 0 \text{ ----- (1)}$$

This is of the form

$$(D - m_1 D' - C_1)(D - m_2 D' - C_2)z = 0 \text{ ----- (2)}$$

Comparing (1) & (2)

$$m_1 = -1, C_1 = 0, m_2 = -1, C_2 = 2.$$

$$C.F = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x)$$

$$C.F = f_1(y - x) + e^{2x} f_2(y - x) \quad \because e^0 = 1$$

To find P.I

$$P.I = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} (e^{3x+y} + \sin(x + 2y))$$



$$P.I = P.I_1 + P.I_2$$

To find P.I₁

$$P.I_1 = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} e^{3x+y} \quad \text{Rule: replace } D = 3 \text{ \& } D' = 1 \text{ Type:1}$$

$$= \frac{1}{9 + 6 + 1 - 6 - 2} e^{3x+y}$$

$$P.I_1 = \frac{1}{8} e^{3x+y}$$

To find P.I₂

$$P.I_2 = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin(x+2y) \quad \text{here } a=1, b=2 \text{ type:2}$$

Rule: replace $D^2 = -(a^2) = -1$; $D'^2 = -(b^2) = -4$ & $DD' = -(ab) = -2$

$$P.I_2 = \frac{1}{-1 + 2(-2) - 4 - 2D - 2D'} \sin(x+2y) = \frac{1}{-1 - 4 - 4 - 2D - 2D'} \sin(x+2y)$$

$$= \frac{-1}{2D + 2D' + 9} \times \frac{D}{D} \sin(x+2y)$$

$$= \frac{-D}{2D^2 + 2D'^2 + 9D} \sin(x+2y)$$

$$= \frac{-D}{-2 - 8 + 9D} \sin(x+2y) = \frac{-D}{9D - 10} \sin(x+2y)$$

If we multiply and divide by D, we can not get the term D^2, D'^2 term, so we take conjugate for constant term and multiplied with both Nr. & Dr.

$$= \frac{-D}{9D - 10} \times \frac{9D + 10}{9D + 10} \sin(x+2y)$$

$$= \frac{-9D^2 - 10}{81D^2 - 100} \sin(x+2y)$$

$$= \frac{-9D^2 \sin(x+2y) - 10D \sin(x+2y)}{-81 - 100}$$

$$= \frac{1}{-181} [-9D \cos(x+2y) - 10 \cos(x+2y)]$$

$$P.I_2 = \frac{1}{181} [9 \sin(x+2y) - 10 \cos(x+2y)]$$

$$P.I = \frac{1}{8} e^{3x+y} + \frac{1}{181} [9 \sin(x+2y) - 10 \cos(x+2y)]$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y-x) + e^{2x} f_2(y-x) + \frac{1}{8} e^{3x+y} + \frac{1}{181} [9 \sin(x+2y) - 10 \cos(x+2y)]$$

2. Solve $(D^2 - D'^2 - 3D + 3D')z = e^{3x+y} + 4$

Solution:

Given $(D^2 - D'^2 - 3D + 3D')z = e^{3x+y} + 4$

To find C.F

$$((D + D')(D - D') - 3(D - D'))z = 0$$

$$(D - D')(D + D' - 3)z = 0 \text{ ----- (1)}$$

This is of the form

$$(D - m_1 D' - C_1)(D - m_2 D' - C_2)z = 0 \text{ ----- (2)}$$

Comparing (1) & (2)



$$m_1 = 1, C_1 = 0, m_2 = -1, C_2 = 3.$$

$$C.F = e^{a_1x} f_1(y + m_1x) + e^{a_2x} f_2(y + m_2x)$$

$$C.F = f_1(y + x) + e^{3x} f_2(y - x) \quad \because e^0 = 1$$

To find P.I

$$P.I = \frac{1}{D^2 - D'^2 - 3D + 3D'} (e^{3x+y} + 4)$$

$$P.I = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2 - D'^2 - 3D + 3D'} e^{3x+y} \quad \text{here } a = 3, b = 1 \quad \text{type : 1}$$

$$= \frac{1}{9 - 1 - 9 + 3} e^{3x+y} \quad \text{Rule: Replace } D = 3, D' = 1$$

$$P.I_1 = \frac{1}{2} e^{3x+y}$$

$$P.I_2 = \frac{1}{D^2 - D'^2 - 3D + 3D'} 4e^{0x+0y} \quad \text{here } a = 0, b = 0 \quad \text{type : 1}$$

$$= \frac{1}{0} 4e^{0x+0y} \quad \text{Rule: Replace } D = 0, D' = 0$$

Introduce x in Nr. and Diff. Dr. Partially w.r.to.D in the previous step

$$= \frac{x}{2D - 0 - 3 + 0} 4e^{0x+0y}$$

$$= \frac{4x}{-3}$$

$$P.I_2 = \frac{-4x}{3}$$

$$P.I = \frac{1}{2} e^{3x+y} - \frac{4x}{3}$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y + x) + e^{3x} f_2(y - x) + \frac{1}{2} e^{3x+y} - \frac{4x}{3}$$

3. Solve $(2D^2 - DD' - D'^2 + 6D + 3D')z = xe^y$

Solution:

Given $(2D^2 - DD' - D'^2 + 6D + 3D')z = xe^y$

To find C.F

$$(2D^2 - DD' - D'^2 + 6D + 3D')z = 0$$

$$(2D + D')(D - D') + 3(2D + D')z = 0$$

$$(2D + D')(D - D' + 3)z = 0$$

$$\left(D + \frac{D'}{2}\right)(D - D' + 3)z = 0$$

This is of the form

$$(D - m_1D' - C_1)(D - m_2D' - C_2)z = 0 \text{----- (2)}$$

Comparing (1) & (2)

$$m_1 = \frac{-1}{2}, C_1 = 0, m_2 = 1, C_2 = -3.$$



$$C.F = e^{c_1x} f_1(y + m_1x) + e^{c_2x} f_2(y + m_2x)$$

$$C.F = f_1\left(y - \frac{x}{2}\right) + e^{-3x} f_2(y + x) \quad \because e^0 = 1$$

To find P.I

$$P.I = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} x e^y \quad \text{Type : 4}$$

$$P.I = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} x e^{0x+y} \quad \text{here } a = 0, b = 1$$

Rule: replace $D = D + a = D + 0 = D$; $D' = D' + b = D' + 1$

$$\begin{aligned} P.I &= e^{0x+y} \frac{1}{2D^2 - D(D'+1) - (D'+1)^2 + 6D + 3(D'+1)} x \\ &= e^y \frac{1}{2D^2 - DD' - D - D'^2 - 2D' - 1 + 6D + 3D' + 3} x \quad \text{Type : 3} \\ &= e^y \frac{1}{2D^2 - DD' + 5D - D'^2 + D' + 2} x \\ &= e^y \frac{1}{2 \left[1 + \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2} \right) \right]} x \end{aligned}$$

[normally we take out highest power term of D in the homogeneous type, but it is not necessary in the non-homogeneous type]

$$\begin{aligned} &= \frac{e^y}{2} \left[1 + \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2} \right) \right]^{-1} x \quad \because D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y} \\ &= \frac{e^y}{2} \left[1 - \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2} \right) + \dots \right] x \quad [\text{neglect the terms } D', \text{ since } D'(x) = 0] \\ &= \frac{e^y}{2} \left[1 - \left(\frac{5D}{2} \right) + \dots \right] x \quad \because D^2(x) = 0 \\ &= \frac{e^y}{2} \left[x - \frac{5(1)}{2} \right] \quad \because D^2(x) = 0 \\ &= \frac{e^y}{2} \left[\frac{2x - 5}{2} \right] \end{aligned}$$

$$P.I = \frac{e^y}{4} [2x - 5]$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y + x) + e^{3x} f_2(y - x) + \frac{e^y}{4} [2x - 5]$$

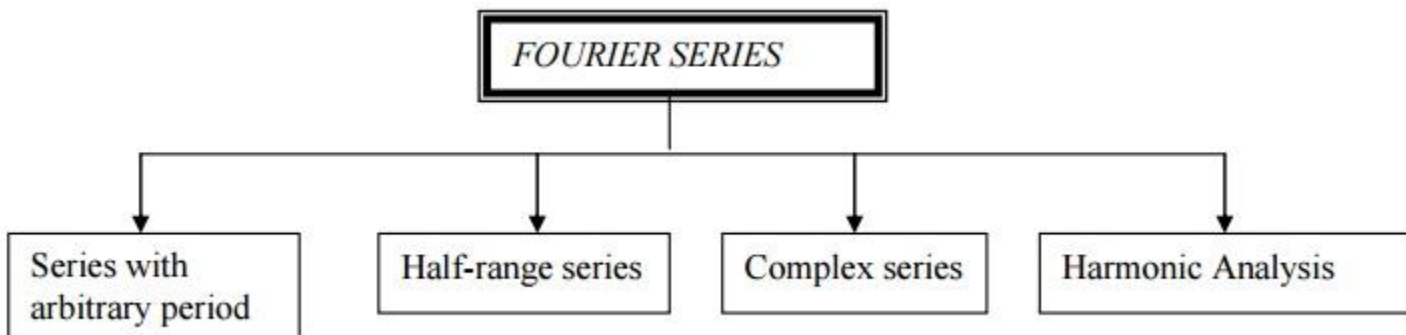


DEPARTMENT OF MATHEMATICS

UNIT II – FOURIER SERIES

2.1 INTRODUCTION OF FOURIER SERIES

A Fourier series of a periodic function consists of a sum of sine and cosine terms. Sines and cosines are the most fundamental periodic functions. The Fourier series is named after the French Mathematician and Physicist Jacques Fourier (1768 –1830). Fourier series has its application in problems pertaining to Heat conduction, acoustics, etc. The subject matter may be divided into the following sub topics.



Convergence of Fourier Series:

- At a continuous point $x = a$, Fourier series converges to $f(a)$
- At end point c or $c+2l$ in $(c, c+2l)$, Fourier series converges to $\frac{f(c) + f(c+2l)}{2}$
- At a discontinuous point $x = a$, Fourier series converges to $\frac{f(a-) + f(a+)}{2}$



Fourier series in an interval of length 2ℓ

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$$

Fourier series of $f(x)$ in $(0, 2\ell)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$$

$$a_0 = \frac{1}{\ell} \int_0^{2\ell} f(x) dx$$

$$a_n = \frac{1}{\ell} \int_0^{2\ell} f(x) \cos \frac{n\pi x}{\ell} dx$$

$$b_n = \frac{1}{\ell} \int_0^{2\ell} f(x) \sin \frac{n\pi x}{\ell} dx$$

Fourier series of $f(x)$ in $(-\ell, \ell)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$$

$$a_0 = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) dx$$

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx$$

Even Function

Odd Function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} \right)$$

$$a_0 = \frac{2}{\ell} \int_0^{\ell} f(x) dx$$

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx$$

$$b_n = 0$$

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi x}{\ell} \right)$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx$$



Fourier series in the Interval of length 2π

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Fourier Series of $f(x)$ in $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx$$

Fourier Series of $f(x)$ in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Even Function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx)$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$b_n = 0$$

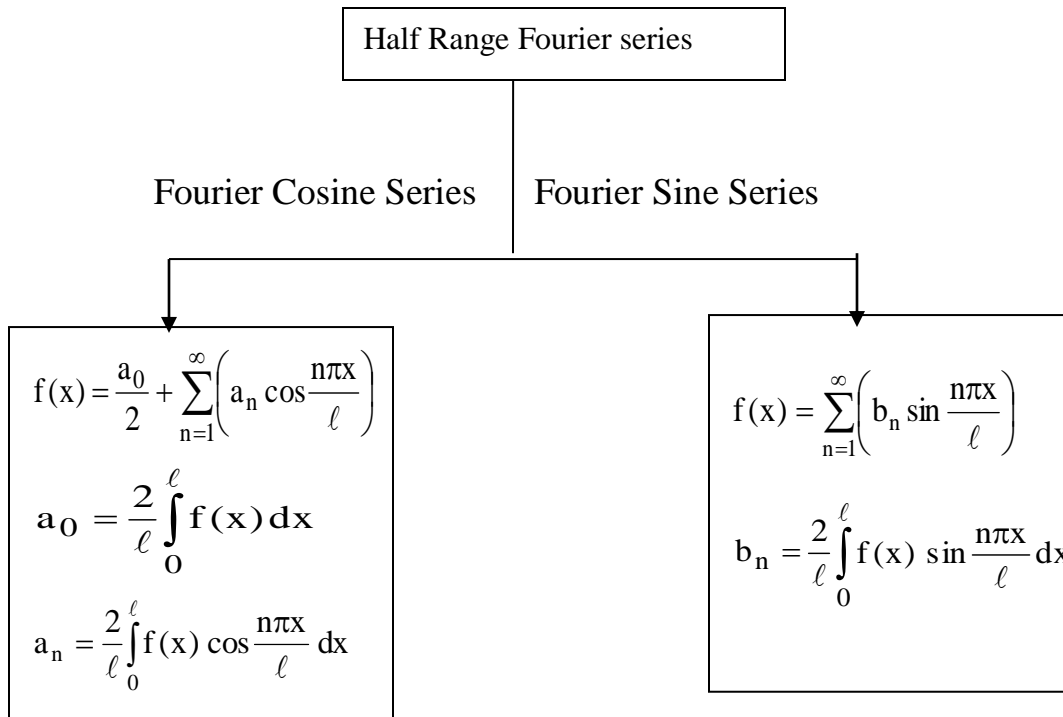
Odd Function

$$f(x) = \sum_{n=1}^{\infty} (b_n \sin nx)$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$



Convergence of Fourier Cosine series:

- At a continuous point $x = a$, Fourier cosine series converges to $f(a)$.
- At end point 0 in $(0, l)$, Fourier cosine series converges to $f(0+)$
- At end point l in $(0, l)$, Fourier cosine series converges to $f(l-)$

Convergence of Fourier Sine series:

- At a continuous point $x = a$, Fourier Sine series converges to $f(a)$.
- At both end points Fourier Sine series converges to 0 .

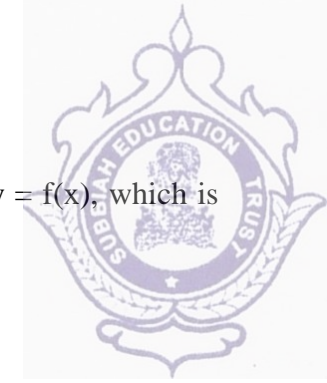
Harmonic Analysis:

$$a_0 = 2 \left[\frac{\sum y}{N} \right], \quad a_n = 2 \left[\frac{\sum y \cos \left(\frac{n\pi x}{l} \right)}{N} \right], \quad b_n = 2 \left[\frac{\sum y \sin \left(\frac{n\pi x}{l} \right)}{N} \right]$$

Parseval's Theorem:

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$ is the Fourier series of $f(x)$ in $(c, c+2l)$,

Then $\bar{y}^2 = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$ (or) $\frac{1}{2l} \int_c^{c+2l} [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$



Root Mean Square Value:

\bar{y}^2 is the effective value (or) Root Mean square (RMS) value of the function $y = f(x)$, which is given by

$$\bar{y} = \sqrt{\frac{\int_c^{c+2\ell} [f(x)]^2 dx}{2\ell}}$$

Some Important Results:

1. $\sin n\pi = 0$ for all integer values of n
2. $\cos n\pi = (-1)^n$ for all integer values of n
3. $\cos 2n\pi = 1$ for all integer values of n
4. $\sin 2n\pi = 0$ for all integer values of n
5. If $f(-x) = f(x)$ then $f(x)$ is even and If $f(-x) = -f(x)$ then $f(x)$ is odd.
6. $f(x) = \begin{cases} \phi_1(x) & (-\ell, 0) \\ \phi_2(x) & (0, \ell) \end{cases}$ is even if either $\phi_1(-x) = \phi_2(x)$ or $\phi_2(-x) = \phi_1(x)$
7. $f(x) = \begin{cases} \phi_1(x) & (-\ell, 0) \\ \phi_2(x) & (0, \ell) \end{cases}$ is odd if either $\phi_1(-x) = -\phi_2(x)$ or $\phi_2(-x) = -\phi_1(x)$
8. $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$
9. $\int e^{ax} \cos bxdx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$
10. $\int e^{ax} \sin bxdx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$
11. $\int u dv = uv_1 - u'v_2 + u''v_3 - \dots$ Where
 $u' = \frac{du}{dx}, u'' = \frac{d^2u}{dx^2}, \dots, v_1 = \int dv, v_2 = \int v_1 dx, v_3 = \int v_2 dx, \dots$



DEPARTMENT OF MATHEMATICS

UNIT - II : FOURIER SERIES

2.2 GENERAL FORM OF FOURIER SERIES

PART A

1. State Dirichlet’s conditions for the existence of Fourier series of $f(x)$ in the interval $(0,2\pi)$.

A function $f(x)$ can be expanded as a Fourier series in the interval $(0,2\pi)$ if the following conditions are satisfied.

- (i) $f(x)$ is periodic, single valued and finite in $(0,2\pi)$
- (ii) $f(x)$ has only finite number of finite discontinuities and no infinite discontinuities in $(0,2\pi)$.
- (iii) $f(x)$ has only finite number of maxima and minima in $(0,2\pi)$

2. Does $f(x) = \tan x$ possess a Fourier expansion?

Solution:

$\tan x$ does not possess a Fourier expansion because the function $f(x) = \tan x = \frac{\sin x}{\cos x}$ has the infinite discontinuity at the point $x = \frac{\pi}{2}$.

3. Determine the value of a_n & a_0 in the Fourier series expansion of $f(x) = x^3$ in $-\pi < x < \pi$

Solution:

$f(x) = x^3 \Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x) \Rightarrow f(x)$ is an odd function $\therefore a_n = a_0 = 0$

4. Find the Fourier constant b_n for $x \sin x$ in $-\pi < x < \pi$, when expressed as a Fourier series.

Solution:

$$f(x) = x \sin x, -\pi < x < \pi$$

$$f(-x) = (-x) \sin(-x) = x \sin x = f(x)$$

$\therefore f(x)$ is an even function $\therefore b_n = 0$

5. Find the constant term of the Fourier series for the function $f(x) = x^2, -\pi < x < \pi$

Solution:

$$f(x) = x^2, -\pi < x < \pi$$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$\therefore f(x)$ is an even function

$$l = \frac{U.L - L.L}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} = \pi$$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{\pi} \int_0^\pi x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^\pi = \frac{2}{\pi} \left[\frac{\pi^3}{3} - 0 \right] = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

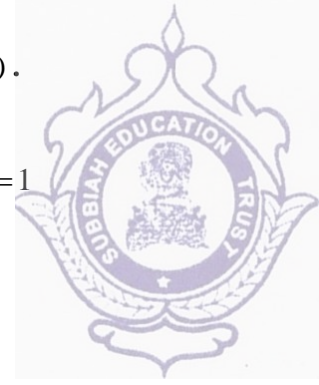
$$\therefore \text{Constant term} = \frac{a_0}{2} = \frac{\pi^2}{3}$$

6. Find the root mean square value of the function $f(x) = x$ in $(0,l)$

$$\text{RMS value} = \bar{y} = \sqrt{\frac{1}{b-a} \int_a^b [f(x)]^2 dx} = \sqrt{\frac{1}{l-0} \int_0^l x^2 dx} = \sqrt{\frac{1}{l} \left[\frac{x^3}{3} \right]_0^l} = \sqrt{\frac{l^3}{3l}} = \frac{l}{\sqrt{3}}$$

7. What do you mean by Harmonic analysis ?

The process of finding the harmonics in the Fourier series expansion of a function numerically is known as harmonic analysis.



8. Find the constant term in the expression of $\cos^2 x$ as a Fourier series in the interval $(-\pi, \pi)$.

Solution:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 x dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1 + \cos 2x}{2} \right) dx = \frac{1}{2\pi} \left[x + \frac{\sin 2x}{2} \right]_{-\pi}^{\pi} = \frac{1}{2\pi} [(\pi) - (-\pi)] = \frac{2\pi}{2\pi} = 1$$

$$\therefore \text{Constant term} = \boxed{\frac{a_0}{2} = \frac{1}{2}}$$

PART B

1. Find the Fourier series $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $0 < x < 2\pi$. Hence show that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$. (ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$.

Solution:

Given $f(x) = \left(\frac{\pi - x}{2}\right)^2 = \frac{1}{4}(\pi - x)^2 = \frac{1}{4}(\pi^2 - 2\pi x + x^2), 0 < x < 2\pi$

General Fourier is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \frac{2\pi}{2} = \pi \quad \therefore \ell = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{----- (1)}$$

To Find a_0 :

$$\begin{aligned} a_0 &= \frac{1}{\ell} \int_0^{2\ell} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4}(\pi^2 - 2\pi x + x^2) dx = \frac{1}{4\pi} \int_0^{2\pi} (\pi^2 - 2\pi x + x^2) dx \\ &= \frac{1}{4\pi} \left[\pi^2 x - \frac{2\pi x^2}{2} + \frac{x^3}{3} \right]_0^{2\pi} = \frac{1}{4\pi} \left[\left(2\pi^3 - 4\pi^3 + \frac{8\pi^3}{3} \right) - (0) \right] \\ &= \frac{\pi^3}{4\pi} \left[-2 + \frac{8}{3} \right] = \frac{\pi^2}{4} \left[\frac{-6 + 8}{3} \right] = \frac{\pi^2}{4} \left[\frac{2}{3} \right] \end{aligned}$$

$$\boxed{a_0 = \frac{\pi^2}{6}}$$

To Find a_n

$$\begin{aligned} a_n &= \frac{1}{\ell} \int_0^{2\ell} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4}(\pi^2 - 2\pi x + x^2) \cos nx dx = \frac{1}{4\pi} \int_0^{2\pi} (\pi^2 - 2\pi x + x^2) \cos nx dx \\ &= \frac{1}{4\pi} \left[(\pi^2 - 2\pi x + x^2) \left(\frac{\sin nx}{n} \right) - (-2\pi + 2x) \left(\frac{-\cos nx}{n^2} \right) + (2) \left(\frac{-\sin nx}{n^3} \right) \right]_0^{2\pi} \\ &= \frac{1}{4\pi} \left[\frac{1}{n^2} (-2\pi + 2x) \cos nx \right]_0^{2\pi} \\ &= \frac{1}{4\pi n^2} [(2\pi \cos 2n\pi) - (-2\pi \cos 0)] \\ a_n &= \frac{1}{4\pi n^2} [2\pi + 2\pi] = \frac{4\pi}{4\pi n^2} \quad \because \cos 2n\pi = 1 \text{ \& } \cos 0 = 1 \end{aligned}$$



$$a_n = \frac{1}{n^2}$$

To find b_n :

$$\begin{aligned}
 b_n &= \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} dx \\
 &= \frac{1}{\ell} \int_0^{2\ell} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4} (\pi^2 - 2\pi x + x^2) \sin nx dx = \frac{1}{4\pi} \int_0^{2\pi} (\pi^2 - 2\pi x + x^2) \sin nx dx \\
 &= \frac{1}{4\pi} \left[(\pi^2 - 2\pi x + x^2) \left(\frac{-\cos nx}{n} \right) - (-2\pi + 2x) \left(\frac{-\sin nx}{n^2} \right) + (2) \left(\frac{\cos nx}{n^3} \right) \right]_0^{2\pi} \\
 &= \frac{1}{4\pi} \left[\frac{-1}{n} (\pi^2 - 2\pi x + x^2) \cos nx + \frac{2}{n^3} \cos nx \right]_0^{2\pi} \\
 &= \frac{1}{4\pi} \left[\left(\frac{-1}{n} (\pi^2 - 4\pi^2 + 4\pi^2) \cos 2n\pi + \frac{2}{n^3} \cos 2n\pi \right) - \left(\frac{-1}{n} \pi^2 \cos 0 + \frac{2}{n^3} \cos 0 \right) \right] \\
 &= \frac{1}{4\pi} \left(\frac{-\pi^2}{n} + \frac{-\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3} \right)
 \end{aligned}$$

$$b_n = 0$$

Substitute a_0, a_n, b_n in (1)

$$\begin{aligned}
 f(x) &= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx + \sum_{n=1}^{\infty} 0 \sin nx \\
 \therefore f(x) &= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx \quad \text{-----(2)}
 \end{aligned}$$

Deduction:

(i) Let $x=0$ be a point of discontinuity

$$(2) \Rightarrow f(0) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos 0 \quad \because f(0) = \frac{f(0) + f(2\pi)}{2} = \frac{\frac{\pi^2}{4} + \frac{\pi^2}{4}}{2} = \frac{2 \frac{\pi^2}{4}}{2} = \frac{\pi^2}{4}$$

$$\because f(x) = \left(\frac{\pi - x}{2} \right)^2$$

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{4\pi^2}{48} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \Rightarrow \boxed{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}}$$

ii) Let $x = \pi$ be a point of continuity

$$f(\pi) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi \quad \because f(x) = \left(\frac{\pi - x}{2} \right)^2 \Rightarrow f(0) = \left(\frac{\pi - \pi}{2} \right)^2 = 0$$



$$0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

2. Obtain the Fourier series for $f(x)$ of period $2l$ and defined as follows $f(x) = \begin{cases} l-x & , 0 < x \leq l \\ 0 & , l \leq x < 2l \end{cases}$. Hence

deduce that (i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$ (ii) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$

Solution: General Fourier is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{l - (-l)}{2} = \frac{2l}{2} = l \quad \therefore \ell = l$

To find a_0

$$a_0 = \frac{1}{l} \int_0^{2l} f(x) dx = \frac{1}{l} \left[\int_0^l (\ell - x) dx + \int_l^{2l} 0 dx \right]$$

$$= \frac{1}{l} \left[lx - \frac{x^2}{2} \right]_0^l = \frac{1}{l} \left[l^2 - \frac{l^2}{2} \right] = \frac{1}{l} \left[\frac{l^2}{2} \right]$$

$$a_0 = \frac{l}{2}$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l} dx = \frac{1}{l} \left[\int_0^l (\ell - x) \cos \frac{n\pi x}{l} dx + \int_l^{2l} 0 dx \right]$$

$$= \frac{1}{l} \left[(\ell - x) \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (-1) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) \right]_0^l$$

$$= \frac{1}{l} \left[\frac{-l^2}{n^2 \pi^2} \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-l^2}{l n^2 \pi^2} [\cos n\pi - \cos 0] = \frac{l}{n^2 \pi^2} [1 - (-1)^n]$$

$$a_n = \begin{cases} \frac{2l}{n^2 \pi^2}, & n=1,3,5,\dots \\ 0 & , n=2,4,6,\dots \end{cases}$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} dx = \frac{1}{l} \left[\int_0^l (\ell - x) \sin \frac{n\pi x}{l} dx + \int_l^{2l} 0 dx \right]$$



$$b_n = \frac{1}{l} \left[(l-x) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (-1) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) \right]_0^l$$

$$= \frac{1}{l} \left[\frac{-l}{n\pi} (l-x) \cos \frac{n\pi x}{l} \right]_0^l = \frac{-l}{ln\pi} [0 - l \cos 0] = \frac{l^2}{ln\pi}$$

$$b_n = \frac{l}{n\pi}$$

Substitute a_0, a_n, b_n in (1)

$$f(x) = \frac{l}{4} + \sum_{n=1,3,5}^{\infty} \frac{2l}{n^2 \pi^2} \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} \frac{l}{n\pi} \sin \frac{n\pi x}{l}$$

$$\therefore f(x) = \frac{l}{4} + \frac{2l}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{l} + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l} \quad \text{-----(2)}$$

Deduction:

i) Let $x = l/2$ be a point of continuity

$$(2) \Rightarrow f(l/2) = \frac{l}{4} + \frac{2l}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \cos \frac{n\pi}{2} + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \quad \because f(x) = \begin{cases} l-x, & 0 < x \leq l \\ 0, & l \leq x < 2l \end{cases}$$

$$\Rightarrow \frac{l}{2} = \frac{l}{4} + \frac{2l}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \cos \frac{n\pi}{2} + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \quad \because f\left(\frac{l}{2}\right) = l - \frac{l}{2} = \frac{l}{2}$$

$$\Rightarrow \frac{l}{2} - \frac{l}{4} = 0 + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \quad \because \cos \frac{n\pi}{2} = 0 \text{ if } n \text{ is odd}$$

$$\Rightarrow \frac{l}{4} = \frac{l}{\pi} \left[\frac{1}{1} \sin \frac{\pi}{2} + \frac{1}{2} \sin \pi + \frac{1}{3} \sin \frac{3\pi}{2} + \dots \right]$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{1}(1) + \frac{1}{2}(0) + \frac{1}{3}(-1) + \dots \quad \because \sin \frac{\pi}{2} = 1; \sin \frac{3\pi}{2} = -1; \sin \frac{5\pi}{2} = 1; \sin \frac{7\pi}{2} = -1 \text{ etc..}$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

i) Let $x = l$ be a point of continuity

$$(2) \Rightarrow f(l) = \frac{l}{4} + \frac{2l}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \cos n\pi + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi \quad \because f(x) = \begin{cases} l-x, & 0 < x \leq l \\ 0, & l \leq x < 2l \end{cases}$$

$$0 = \frac{l}{4} + \frac{2l}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} (-1)^n + 0 \quad \Rightarrow f(l) = l - l = 0$$

$$-\frac{l}{4} = \frac{2l}{\pi^2} \left[-\frac{1}{1^2} + \frac{1}{3^2} - \frac{1}{5^2} + \dots \right]$$

$$-\frac{\pi^2}{8} = - \left[\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \dots \right]$$

$$\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \dots = \frac{\pi^2}{8}$$

3. Find the Fourier series expansion of $f(x) = \begin{cases} x, & 0 \leq x \leq 3 \\ 6-x, & 3 \leq x \leq 6 \end{cases}$ and hence find the value



of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Solution:

Given $f(x) = \begin{cases} x, & 0 \leq x \leq 3 \\ 6-x, & 3 \leq x \leq 6 \end{cases}$

General Fourier is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{6-0}{2} = 3 \quad \therefore \ell = 3$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3} \quad \text{----- (1)}$

$a_0 = \frac{1}{\ell} \int_0^{2\ell} f(x) dx = \frac{1}{3} \int_0^6 f(x) dx = \frac{1}{3} \left[\int_0^3 x dx + \int_3^6 (6-x) dx \right]$

$= \frac{1}{3} \left\{ \left[\frac{x^2}{2} \right]_0^3 + \left[\frac{(6-x)^2}{-2} \right]_3^6 \right\}$

$= \frac{1}{3} \left\{ \left[\left\{ \frac{3^2}{2} \right\} - \{0\} \right] + \left[\{0\} - \left\{ \frac{(3)^2}{-2} \right\} \right] \right\} = \frac{1}{3} \left[\frac{9}{2} + \frac{9}{2} \right] = \frac{9}{3}$

$a_0 = 3$

$a_n = \frac{1}{\ell} \int_0^{2\ell} f(x) \cos \frac{n\pi x}{\ell} dx$

$= \frac{1}{3} \int_0^6 f(x) \cos \frac{n\pi x}{3} dx$

$= \frac{1}{3} \left[\int_0^3 x \cos \frac{n\pi x}{3} dx + \int_3^6 (6-x) \cos \frac{n\pi x}{3} dx \right]$

$= \frac{1}{3} \left\{ \left[(x) \left(\frac{\sin \frac{n\pi x}{3}}{\frac{n\pi}{3}} \right) - (1) \left(\frac{-\cos \frac{n\pi x}{3}}{\frac{n^2 \pi^2}{9}} \right) \right]_0^3 + \left[(6-x) \left(\frac{\sin \frac{n\pi x}{3}}{\frac{n\pi}{3}} \right) - (-1) \left(\frac{-\cos \frac{n\pi x}{3}}{\frac{n^2 \pi^2}{9}} \right) \right]_3^6 \right\}$

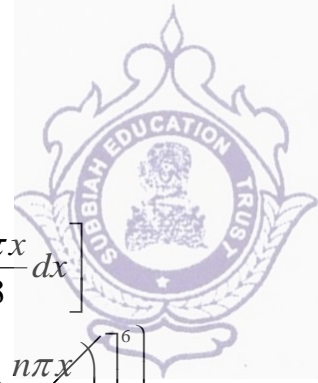
$= \frac{1}{3} \left\{ \left[\frac{9}{n^2 \pi^2} \cos \frac{n\pi x}{3} \right]_0^3 - \left[\frac{9}{n^2 \pi^2} \cos \frac{n\pi x}{3} \right]_3^6 \right\}$

$= \frac{9}{3n^2 \pi^2} \left\{ \left[\cos \frac{n\pi x}{3} \right]_0^3 - \left[\cos \frac{n\pi x}{3} \right]_3^6 \right\}$

$= \frac{3}{n^2 \pi^2} \{ [\cos n\pi - \cos 0] - [\cos 2n\pi - \cos n\pi] \}$

$= \frac{3}{n^2 \pi^2} [\cos n\pi - 1 - 1 + \cos n\pi] = \frac{3}{n^2 \pi^2} [2(-1)^n - 2] = \frac{6}{n^2 \pi^2} [(-1)^n - 1]$

$= \frac{6}{n^2 \pi^2} \begin{cases} -2, & n=1,3,5,\dots \\ 0, & n=2,4,6,\dots \end{cases}$



$$a_n = \begin{cases} \frac{-12}{n^2 \pi^2}, & n=1,3,5,\dots \\ 0, & n=2,4,6,\dots \end{cases}$$

$$\begin{aligned} b_n &= \frac{1}{\ell} \int_0^{2\ell} f(x) \sin \frac{n\pi x}{\ell} dx = \frac{1}{3} \int_0^6 f(x) \sin \frac{n\pi x}{3} dx = \frac{1}{3} \left[\int_0^3 x \sin \frac{n\pi x}{3} dx + \int_3^6 (6-x) \sin \frac{n\pi x}{3} dx \right] \\ &= \frac{1}{3} \left\{ \left[(x) \left(\frac{-\cos \frac{n\pi x}{3}}{\frac{n\pi}{3}} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{3}}{\frac{n^2 \pi^2}{9}} \right) \right]_0^3 + \left[(6-x) \left(\frac{-\cos \frac{n\pi x}{3}}{\frac{n\pi}{3}} \right) - (-1) \left(\frac{-\sin \frac{n\pi x}{3}}{\frac{n^2 \pi^2}{9}} \right) \right]_3^6 \right\} \\ &= \frac{1}{3} \left\{ \left[\frac{-3}{n\pi} x \cos \frac{n\pi x}{3} \right]_0^3 + \left[\frac{-3}{n\pi} (6-x) \cos \frac{n\pi x}{3} \right]_3^6 \right\} \\ &= \frac{-3}{3n\pi} \{ [3 \cos n\pi - 0] + [0 - 3 \cos n\pi] \} = \frac{-3}{3n\pi} [3 \cos n\pi - 3 \cos n\pi] \\ \therefore \boxed{b_n = 0} \end{aligned}$$

Substitute a_0, a_n, b_n in (1)

$$f(x) = \frac{3}{2} + \sum_{n=1,3,5}^{\infty} \frac{-12}{n^2 \pi^2} \cos \frac{n\pi x}{3} + \sum_{n=1}^{\infty} 0 \sin \frac{n\pi x}{3}$$

$$\therefore \boxed{f(x) = \frac{3}{2} - \frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{3}} \text{-----(2)}$$

Deduction:

Let $x=0$ be a point of continuity

$$f(0) = \frac{3}{2} - \frac{12}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \cos 0 \quad f(x) = \begin{cases} x, & 0 \leq x \leq 3 \\ 6-x, & 3 \leq x \leq 6 \end{cases}$$

$$\Rightarrow 0 = \frac{3}{2} - \frac{12}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{12}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} = \frac{3}{2}$$

$$\therefore \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8} \Rightarrow \boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}}$$

4. Find the Fourier series for $f(x) = |x|$ in $-\pi < x < \pi$ and deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$.

Solution:

Given $f(x) = |x|, \quad -\pi < x < \pi$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} \quad \therefore \ell = \pi$

Now, $f(-x) = |-x| = |x| = f(x)$

$\therefore f(x)$ is an even function.

$\therefore \boxed{b_n = 0}$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx \text{ ----- (1) } \quad \because l = \pi$$

To Find a₀:

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{\pi} \int_0^{\pi} |x| dx = \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left(\frac{x^2}{2} \right)_0^{\pi} = \frac{2}{2\pi} (\pi^2 - 0) = \pi$$

$$\therefore \boxed{a_0 = \pi}$$

To Find a_n:

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx = \frac{2}{\pi} \int_0^{\pi} |x| \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$$

$$= \frac{2}{\pi} \left[(x) \left(\frac{\sin nx}{n} \right) - (1) \left(\frac{-\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{1}{n^2} \right) \cos nx \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left(\frac{1}{n^2} \right) [\cos n\pi - \cos 0]$$

$$= \frac{2}{\pi n^2} [(-1)^n - 1]$$

$$a_n = \begin{cases} \frac{-4}{\pi n^2}, & \text{if } n=1,3,5,\dots \\ 0, & \text{if } n=2,4,6,\dots \end{cases}$$

$$\therefore (1) \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1,3,5,\dots}^{\infty} \frac{-4}{n^2 \pi} \cos nx$$

$$\therefore \boxed{f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos nx} \text{ ----- (2)}$$

Deduction:

Let $x = 0$ be a point of continuity

$$(2) \Rightarrow f(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos 0$$

$$\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \quad f(x) = |x| \Rightarrow f(0) = 0$$

$$\Rightarrow \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} = \frac{\pi}{2}$$

$$\sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$



$$\therefore 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

5. Find the Fourier series for $f(x) = x^2$ in $-\pi \leq x \leq \pi$ and deduce that

(a) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$.

(b) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} = \frac{\pi^2}{12}$.

(c) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$

Solution:

Given $f(x) = x^2, -\pi \leq x \leq \pi$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} = \pi \quad \therefore \ell = \pi$

Now, $f(-x) = x^2 = (-x)^2 = x^2 = f(x)$

$\therefore f(x)$ is an even function.

$\therefore b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{----- (1)} \quad \because \ell = \pi$

To Find a_0 :

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2}{3\pi} [\pi^3 - 0] = \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

To find a_n :

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

$$= \frac{2}{\pi} \left[(x^2) \left(\frac{\sin nx}{n} \right) - (2x) \left(\frac{-\cos nx}{n^2} \right) + (2) \left(\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{2}{n^2} \right) x \cos nx \right]_0^{\pi}$$

$$= \frac{4}{\pi n^2} (\pi \cos n\pi - 0)$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$\therefore (1) \Rightarrow f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$



$$\therefore f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \quad \text{----- (2)}$$

Deduction: a

Let $x = \pi$ be a point of continuity

$$\begin{aligned} (2) \Rightarrow f(\pi) &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \pi \\ \Rightarrow \pi^2 &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n^2} && \because f(x) = x^2 \Rightarrow f(\pi) = \pi^2 \\ \Rightarrow \pi^2 - \frac{\pi^2}{3} &= 4 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} \\ \Rightarrow \frac{2\pi^2}{3} &= 4 \sum_{n=1}^{\infty} \frac{1}{n^2} && \because (-1)^{2n} = 1 \end{aligned}$$

$$\therefore 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

Deduction: b

Let $x = 0$ be a point of continuity

$$\begin{aligned} (2) \Rightarrow f(0) &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos 0 \\ \Rightarrow 0 &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} && \because f(x) = x^2 \Rightarrow f(0) = 0 \\ \Rightarrow -\frac{\pi^2}{3} &= 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \\ \Rightarrow -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} &= -\frac{\pi^2}{12} \end{aligned}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} = \frac{\pi^2}{12}$$

Deduction: c

By Parsevals identity for Fourier series,

$$\begin{aligned} \frac{2}{l} \int_0^l [f(x)]^2 dx &= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \\ \frac{2}{\pi} \int_0^{\pi} [x^2]^2 dx &= \frac{4\pi^4}{2} + \sum_{n=1}^{\infty} \left(\frac{4(-1)^n}{n^2} \right)^2 \\ \frac{2}{\pi} \int_0^{\pi} x^4 dx &= \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^4} \\ \frac{2}{\pi} \left[\left(\frac{\pi^5}{5} \right) - (0) \right] &= \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4} && \because (-1)^{2n} = 1 \\ \frac{2\pi^5}{5\pi} &= \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4} \end{aligned}$$



$$\frac{2\pi^4}{5} - \frac{2\pi^4}{9} = 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$2\pi^4 \left(\frac{1}{5} - \frac{1}{9} \right) = 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$2\pi^4 \times \frac{4}{45} = 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{8\pi^4}{45 \times 16} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$$

Obtain the Fourier series of $f(x) = x + x^2$ in $-\pi < x < \pi$. Hence show that

6.

i) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$

Solution:

Given $f(x) = x + x^2$ in $-\pi \leq x \leq \pi$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} = \pi \quad \therefore \ell = \pi$

Now, $f(-x) = -x + (-x)^2$
 $= -x + x^2 \neq f(x)$
 $= -(x - x^2) \neq -f(x)$

$\therefore f(-x) \neq f(x)$ & $f(-x) \neq -f(x)$

$\therefore f(x)$ is Neither even Nor odd function.

General Fourier is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ----- (1) $\because \ell = \pi$

To Find a_0 :

$$a_0 = \frac{1}{\ell} \int_{-l}^l f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) dx = \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left\{ \left(\frac{\pi^2}{2} + \frac{\pi^3}{3} \right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3} \right) \right\} = \frac{1}{\pi} \left\{ \frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3} \right\} = \frac{1}{\pi} \left\{ \frac{2\pi^3}{3} \right\} = \frac{2\pi^2}{3}$$

$$a_0 = \frac{2\pi^2}{3}$$

To Find a_n :

$$a_n = \frac{1}{\ell} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx dx$$



$$\begin{aligned}
 a_n &= \frac{1}{\pi} \left[(x+x^2) \left(\frac{\sin nx}{n} \right) - (1+2x) \left(\frac{-\cos nx}{n^2} \right) + (2) \left(\frac{\sin nx}{n^3} \right) \right]_{-\pi}^{\pi} \\
 &= \frac{1}{\pi} \left[\left(\frac{1}{n^2} \right) (1+2x) \cos nx \right]_{-\pi}^{\pi} \\
 &= \frac{1}{\pi n^2} \{ [(1+2\pi) \cos n\pi] - [(1-2\pi) \cos(-n\pi)] \} \\
 &= \frac{1}{\pi n^2} [(-1)^n (\cancel{\lambda} + 2\pi - \cancel{\lambda} + 2\pi)] \quad \because \cos(-n\pi) = \cos n\pi = (-1)^n \\
 &= \frac{(-1)^n}{n^2 \pi} [4\pi] = \frac{4(-1)^n}{n^2}
 \end{aligned}$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

To Find b_n :

$$\begin{aligned}
 b_n &= \frac{1}{\ell} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} (x+x^2) \sin nx dx \\
 &= \frac{1}{\pi} \left[(x+x^2) \left(\frac{-\cos nx}{n} \right) - (1+2x) \left(\frac{-\sin nx}{n^2} \right) + (2) \left(\frac{\cos nx}{n^3} \right) \right]_{-\pi}^{\pi} \\
 &= \frac{1}{\pi} \left[\left(\frac{-1}{n} \right) (x+x^2) \cos nx + \left(\frac{2}{n^3} \right) \cos nx \right]_{-\pi}^{\pi} \quad \because \cos(-n\pi) = \cos n\pi = (-1)^n \\
 &= \frac{1}{\pi} \left\{ \left[\left(\frac{-1}{n} \right) (\pi + \pi^2) \cos n\pi + \left(\frac{2}{n^3} \right) \cos n\pi \right] - \left[\left(\frac{-1}{n} \right) (-\pi + \pi^2) \cos n\pi + \left(\frac{2}{n^3} \right) \cos(-n\pi) \right] \right\} \\
 &= \frac{(-1)^n}{\pi} \left(-\frac{\pi}{n} - \frac{\pi^2}{n} + \frac{2}{n^3} - \frac{\pi}{n} + \frac{\pi^2}{n} - \frac{2}{n^3} \right) = -\frac{2\pi(-1)^n}{n\pi}
 \end{aligned}$$

$$\therefore \boxed{b_n = \frac{-2}{n} (-1)^n}$$

$$(1) \Rightarrow f(x) = \frac{\pi^2}{3} + \sum_1^{\infty} \frac{4(-1)^n}{n^2} \cos nx + \sum_1^{\infty} \frac{-2(-1)^n}{n} \sin nx$$

$$\boxed{f(x) = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^n}{n^2} \cos nx - 2 \sum_1^{\infty} \frac{(-1)^n}{n} \sin nx} \quad \text{----- (2)}$$

Deduction: 1

Let $x = \pi$ be the point of discontinuity



$$(2) \Rightarrow f(\pi) = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^n}{n^2} \cos n\pi - 2 \sum_1^{\infty} \frac{(-1)^n}{n} \sin n\pi$$

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{1}{n^2} \quad \because f(x) = x + x^2 \Rightarrow f(\pi) = \frac{f(-\pi) + f(\pi)}{2} = \frac{-\pi + \pi^2 + \pi + \pi^2}{2} = \frac{2\pi^2}{2} = \pi^2$$

$$\pi^2 - \frac{\pi^2}{3} = 4 \sum_1^{\infty} \frac{1}{n^2}$$

$$\frac{2\pi^2}{3 \times 4} = \sum_1^{\infty} \frac{1}{n^2}$$

$$\boxed{\sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

Deduction: 2

Let $x=0$ be the point of continuity

$$(2) \Rightarrow f(0) = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^n}{n^2} \cos 0 - 2 \sum_1^{\infty} \frac{(-1)^n}{n} \sin 0$$

$$0 = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^n}{n^2} \quad f(x) = x + x^2 \Rightarrow f(0) = 0$$

$$-\frac{\pi^2}{3} = 4 \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^3} \dots \right]$$

$$\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^3} - \dots = \frac{\pi^2}{12}}$$

7. Find the Fourier series expansion of $f(x)$ where $f(x) = \begin{cases} \pi + x, & -\pi \leq x \leq 0 \\ \pi - x, & 0 \leq x \leq \pi \end{cases}$ and hence deduce that

$$\sum_1^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Solution:

$$\text{Given } f(x) = \begin{cases} \pi + x, & -\pi \leq x \leq 0 \\ \pi - x, & 0 \leq x \leq \pi \end{cases}$$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} = \pi \quad \therefore \ell = \pi$$

$$\text{Let } f(x) = \begin{cases} f_1(x), & -\pi \leq x \leq 0 \\ f_2(x), & 0 \leq x \leq \pi \end{cases}$$

Where

$$f_1(x) = \pi + x \quad f_2(x) = \pi - x$$

$$f_1(-x) = \pi - x = f_2(x) \quad f_2(-x) = \pi + x = f_1(x)$$

$\therefore f(x)$ is an even function.

$$\therefore \boxed{b_n = 0}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

$$\therefore \boxed{f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx} \text{ ----- (1) } \quad \because \ell = \pi$$



To Find a_0 :

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx = \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left(\pi x - \frac{x^2}{2} \right)_0^{\pi} = \frac{2}{\pi} \left(\pi^2 - \frac{\pi^2}{2} \right) = \frac{2}{\pi} \times \frac{\pi^2}{2} = \pi$$

$$\therefore \boxed{a_0 = \pi}$$

To Find a_n :

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos nx dx$$

$$= \frac{2}{\pi} \left[(\pi - x) \left(\frac{\sin nx}{n} \right) - (-1) \left(\frac{-\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{-1}{n^2} \right) \cos nx \right]_0^{\pi}$$

$$= \frac{-2}{n^2 \pi} [\cos n\pi - \cos 0]$$

$$= \frac{-2}{\pi n^2} [(-1)^n - 1]$$

$$a_n = \begin{cases} \frac{4}{\pi n^2}, & \text{if } n=1,3,5,\dots \\ 0, & \text{if } n=2,4,6,\dots \end{cases}$$

$$\therefore (1) \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1,3,5,\dots}^{\infty} \frac{4}{n^2 \pi} \cos nx$$

$$\therefore \boxed{f(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos nx} \text{----- (2)}$$

Deduction:

Let $x = 0$ be a point of continuity

$$(2) \Rightarrow f(0) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos 0$$

$$\Rightarrow \pi = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2}$$

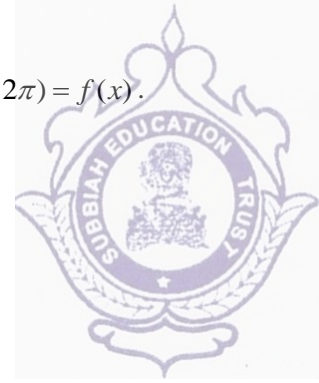
$$\therefore f(x) = \begin{cases} \pi + x, & -\pi \leq x \leq 0 \\ \pi - x, & 0 \leq x \leq \pi \end{cases} \Rightarrow f(0) = \pi$$

$$\Rightarrow \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} = \pi - \frac{\pi}{2}$$

$$\sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} = \frac{\pi}{2} \times \frac{\pi}{4} = \frac{\pi^2}{8}$$

$$\therefore \boxed{\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}}$$

$$\therefore \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$



8. Determine the Fourier series expansion of $f(x)$ where $f(x) = \begin{cases} -1+x, & -\pi \leq x \leq 0 \\ 1+x, & 0 \leq x \leq \pi \end{cases}$ with $f(x+2\pi) = f(x)$.

Solution:

$$\text{Given } f(x) = \begin{cases} -1+x, & -\pi \leq x \leq 0 \\ 1+x, & 0 \leq x \leq \pi \end{cases}$$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} = \pi \quad \therefore \ell = \pi$$

$$\text{Let } f(x) = \begin{cases} f_1(x), & -\pi \leq x \leq 0 \\ f_2(x), & 0 \leq x \leq \pi \end{cases}$$

Where

$$f_1(x) = -1+x$$

$$f_2(x) = 1+x$$

$$f_1(-x) = -1-x = -(1+x) = -f_2(x)$$

$$f_2(-x) = 1-x = -(-1+x) = -f_1(x)$$

$\therefore f(x)$ is an odd function.

$$\therefore a_0 = a_n = 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{----- (1)} \quad \because \ell = \pi$$

To Find b_n :

$$b_n = \frac{1}{\ell} \int_{-l}^l f(x) \sin \frac{n\pi x}{\ell} dx = \frac{2}{\pi} \int_0^{\pi} (1+x) \sin nx dx$$

$$= \frac{2}{\pi} \left[(1+x) \left(\frac{-\cos nx}{n} \right) - (1) \left(\frac{-\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{-1}{n} \right) (1+x) \cos nx \right]_0^{\pi}$$

$$= \frac{-2}{n\pi} [(1+\pi) \cos n\pi - (1+0) \cos 0]$$

$$= \frac{-2}{n\pi} [(1+\pi)(-1)^n - 1]$$

$$b_n = \frac{2}{n\pi} [1 - (1+\pi)(-1)^n]$$

$$\therefore (1) \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - (1+\pi)(-1)^n] \sin nx$$

$$\therefore f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [1 - (1+\pi)(-1)^n] \sin nx$$

9. Find the Fourier series for $f(x)$ where $f(x) = \begin{cases} 0, & 1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$.

Solution:



$$\text{Given } f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

$$\text{Here } l = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{1 - (-1)}{2} = \frac{2}{2} = 1 \quad \therefore l = 1$$

$$\text{Let } f(x) = \begin{cases} f_1(x), & -1 \leq x \leq 0 \\ f_2(x), & 0 \leq x \leq 1 \end{cases}$$

Where

$$\begin{aligned} f_1(x) &= 0 & f_2(x) &= 1 \\ f_1(-x) &= 0 \neq f_2(x) & f_2(-x) &= 1 \neq f_1(x) \\ f_1(-x) &= 0 \neq -f_2(x) & f_2(-x) &= 1 \neq -f_1(x) \end{aligned}$$

$\therefore f(x)$ is Neither even Nor odd function.

$$\text{General Fourier is } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\therefore \boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x} \text{ ----- (1) } \quad \because l = 1$$

To Find a_0 :

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx = \frac{1}{1} \int_{-1}^1 f(x) dx = \int_{-1}^0 0 dx + \int_0^1 1 dx = [x]_0^1 = 1 - 0 = 1$$

$$\boxed{\therefore a_0 = 1}$$

To Find a_n :

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx$$

$$= \frac{1}{1} \int_{-1}^1 f(x) \cos \frac{n\pi x}{1} dx$$

$$= \int_{-1}^0 0 dx + \int_0^1 1 \cos n\pi x dx$$

$$= \left[\frac{\sin n\pi x}{n\pi} \right]_0^1$$

$$= [\sin n\pi - \sin 0] = 0$$

$$\boxed{a_n = 0}$$

To Find b_n :

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{1}{1} \int_{-1}^1 f(x) \sin \frac{n\pi x}{1} dx$$

$$= \int_{-1}^0 0 dx + \int_0^1 1 \sin n\pi x dx$$

$$= \left[\frac{-\cos n\pi x}{n\pi} \right]_0^1$$



$$\begin{aligned}
 &= \frac{-1}{n\pi} [\cos n\pi x]_0^1 \\
 &= \frac{-1}{n\pi} [\cos n\pi - \cos 0]_0^1 \\
 &= \frac{1}{n\pi} [1 - (-1)^n]
 \end{aligned}$$

$$b_n = \begin{cases} \frac{2}{n\pi}, & \text{if } n = 1, 3, 5, \dots \\ 0, & \text{if } n = 2, 4, 6, \dots \end{cases}$$

$$(1) \Rightarrow f(x) = \frac{1}{2} + \sum_{1,3}^{\infty} \frac{2}{n\pi} \sin n\pi x.$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{1,3}^{\infty} \frac{1}{n} \sin n\pi x$$

10. Find the Fourier series for $f(x) = |\cos x|$ in the interval $(-\pi, \pi)$.

Solution:

Given $f(x) = |\cos x|, -\pi < x < \pi$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} \therefore \ell = \pi$

Now, $f(-x) = |\cos(-x)| = |\cos x| = f(x)$

$\therefore f(x)$ is an even function.

$\therefore b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx \text{ ----- (1) } \quad \because \ell = \pi$$

To find a_0 :

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} |\cos x| dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} -\cos x dx \right] \quad \because |\cos x| = \begin{cases} \cos x, & 0 < x < \frac{\pi}{2} \\ -\cos x, & \frac{\pi}{2} < x < \pi \end{cases}$$

$$= \frac{2}{\pi} \left[(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{\pi} \left[\left(\sin \frac{\pi}{2} - \sin 0 \right) - \left(\sin \pi - \sin \frac{\pi}{2} \right) \right] = \frac{2}{\pi} (1 - (-1))$$

$\therefore a_0 = \frac{4}{\pi}$

To find a_n :



$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx$$

$$a_n = \frac{2}{\pi} \int_0^\pi |\cos x| \cos nx dx$$

$$a_n = \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \cos nx dx + \int_{\pi/2}^\pi -\cos x \cos nx dx \right]$$

$$a_n = \frac{2}{\pi} \left[\int_0^{\pi/2} \cos nx \cos x dx - \int_{\pi/2}^\pi \cos nx \cos x dx \right]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)] \quad \text{Here } A = nx \quad B = x$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\int_0^{\pi/2} \frac{1}{2} [\cos(n+1)x + \cos(n-1)x] dx - \int_{\pi/2}^\pi \frac{1}{2} [\cos(n+1)x + \cos(n-1)x] dx \right] \\ &= \frac{1}{\pi} \left[\int_0^{\pi/2} [\cos(n+1)x + \cos(n-1)x] dx - \int_{\pi/2}^\pi [\cos(n+1)x + \cos(n-1)x] dx \right] \\ &= \frac{1}{\pi} \left\{ \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_0^{\pi/2} - \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_{\pi/2}^\pi \right\} \\ &= \frac{1}{\pi} \left[\left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} \right) - (0) - \left(0 - \left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} \right) \right) \right] \\ &= \frac{1}{\pi} \left(\frac{\sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right)}{n+1} + \frac{\sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right)}{n-1} + \frac{\sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right)}{n+1} + \frac{\sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right)}{n-1} \right) \end{aligned}$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right) = \sin \frac{n\pi}{2} \cos \frac{\pi}{2} + \cos \frac{n\pi}{2} \sin \frac{\pi}{2} = \cos \frac{n\pi}{2} \quad \because \cos \frac{\pi}{2} = 0 \& \sin \frac{\pi}{2} = 1$$

$$\sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right) = \sin \frac{n\pi}{2} \cos \frac{\pi}{2} - \cos \frac{n\pi}{2} \sin \frac{\pi}{2} = -\cos \frac{n\pi}{2} \quad \because \cos \frac{\pi}{2} = 0 \& \sin \frac{\pi}{2} = 1$$

$$a_n = \frac{1}{\pi} \left(\frac{\cos \frac{n\pi}{2}}{n+1} - \frac{\cos \frac{n\pi}{2}}{n-1} + \frac{\cos \frac{n\pi}{2}}{n+1} - \frac{\cos \frac{n\pi}{2}}{n-1} \right)$$

$$= \frac{2}{\pi} \cos \frac{n\pi}{2} \left(\frac{1}{n+1} - \frac{1}{n-1} \right)$$

$$= \frac{2}{\pi} \cos \frac{n\pi}{2} \left(\frac{n-1-n-1}{(n+1)(n-1)} \right)$$

$$a_n = \frac{2}{\pi} \cos \frac{n\pi}{2} \left(\frac{-2}{n^2-1} \right)$$



$$a_n = \frac{-4}{\pi(n^2 - 1)} \cos \frac{n\pi}{2}, \text{ Provided } n \neq 1$$

When $n = 1$

$$\begin{aligned} a_1 &= \frac{2}{\pi} \int_0^\pi |\cos x| \cos x dx \\ &= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \cos x dx + \int_{\pi/2}^\pi -\cos x \cos x dx \right] \\ &= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos^2 x dx - \int_{\pi/2}^\pi \cos^2 x dx \right] \\ &= \frac{2}{\pi} \left[\int_0^{\pi/2} \left(\frac{1 + \cos 2x}{2} \right) dx - \int_{\pi/2}^\pi \left(\frac{1 + \cos 2x}{2} \right) dx \right] \\ &= \frac{2}{2\pi} \left[\int_0^{\pi/2} (1 + \cos 2x) dx - \int_{\pi/2}^\pi (1 + \cos 2x) dx \right] \\ &= \frac{1}{\pi} \left[\left(x + \frac{\sin 2x}{2} \right)_0^{\pi/2} - \left(x + \frac{\sin 2x}{2} \right)_{\pi/2}^\pi \right] \\ &= \frac{1}{\pi} \left\{ \left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) - (0) \right] - \left[\left(\pi + \frac{\sin 2\pi}{2} \right) - \left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) \right] \right\} \\ a_1 &= \frac{1}{\pi} \left[\frac{\pi}{2} - \pi + \frac{\pi}{2} \right] \end{aligned}$$

$$a_1 = 0$$

$$\therefore f(x) = \frac{4}{\pi} + \sum_1^\infty \frac{-4}{\pi(n^2 - 1)} \cos \frac{n\pi}{2} \cos nx$$

$$\therefore f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_1^\infty \frac{1}{(n^2 - 1)} \cos \frac{n\pi}{2} \cos nx$$

11. Find the half range Fourier sine series for $f(x) = x(\pi - x)$ in the interval $(0, \pi)$ and deduce that

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots \infty$$

Solution:

Given $f(x) = x(\pi - x) = \pi x - x^2, \quad (0, \pi)$

: General Fourier is $f(x) = \sum_{n=1}^\infty b_n \sin \frac{n\pi x}{\ell}$

Here $\ell = \text{Upper Limit} - \text{Lower Limit} = \pi - 0 = \pi \quad \therefore \ell = \pi$

$$f(x) = \sum_{n=1}^\infty b_n \sin nx \quad \text{----- (1)}$$

To Find b_n :

$$\begin{aligned} b_n &= \frac{2}{\ell} \int_0^\ell f(x) \sin \frac{n\pi x}{\ell} dx \\ &= \frac{2}{\ell} \int_0^\ell x(\pi - x) \sin nx dx \end{aligned}$$



$$= \frac{2}{\ell} \int_0^{\ell} (\pi x - x^2) \sin nx \, dx$$

$$b_n = \frac{2}{\pi} \left[(\pi x - x^2) \left(\frac{-\cos nx}{n} \right) - (\pi - 2x) \left(\frac{-\sin nx}{n^2} \right) + (-2) \left(\frac{\cos nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\frac{2}{n^3} \cos nx \right]_0^{\pi}$$

$$= \frac{-4}{\pi n^3} [\cos n\pi - \cos 0]$$

$$= \frac{-4}{\pi n^3} [(-1)^n - 1]$$

$$b_n = \begin{cases} \frac{8}{\pi n^3}, & \text{if } n=1,3,5,\dots \\ 0, & \text{if } n=2,4,6,\dots \end{cases}$$

The required Fourier sine series be

$$(1) \Rightarrow f(x) = \sum_{n=1,3,5,\dots}^{\infty} \frac{8}{\pi n^3} \sin nx$$

$$f(x) = \frac{8}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin nx \quad \text{----- (2)}$$

Deduction:

Let $x = \frac{\pi}{2}$ be a point of continuity.

$$(2) \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{8}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi}{2}\right) \quad \because f(x) = x(\pi - x) \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\pi^2}{4} = \frac{2\pi^2 - \pi^2}{4}$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{8}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi}{2}\right) \quad \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

$$\Rightarrow \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi}{2}\right) = \frac{\pi^3}{32}$$

$$\Rightarrow \frac{1}{1^3} \sin \frac{\pi}{2} + \frac{1}{3^3} \sin \frac{\pi}{2} + \frac{1}{5^3} \sin \frac{\pi}{2} + \dots = \frac{\pi^3}{32}$$

$$\Rightarrow \boxed{\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32}}$$

12. Find the half – range cosine series for $f(x) = (x - 1)^2$ in $(0, 1)$. Hence show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$.

Solution:

Given:

Here $\ell = \text{Upper Limit} - \text{Lower Limit} = 1 - 0 = 1 \quad \therefore \ell = 1$

Let the cosine series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \quad \because l=1 \text{----- (1)}$$

To Find a_0 :

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{1} \int_0^1 (x^2 - 2x + 1) dx = 2 \left[\frac{x^3}{3} - 2 \frac{x^2}{2} + x \right]_0^1 = 2 \left(\frac{1}{3} - 1 + 1 \right) = \frac{2}{3}$$

$$\boxed{a_0 = \frac{2}{3}}$$

To Find a_n :

$$a_n = 2 \int_0^1 (x^2 - 2x + 1) \cos n\pi x dx$$

$$= 2 \left[(x^2 - 2x + 1) \left(\frac{\sin n\pi x}{n\pi} \right) - (2x - 2) \left(\frac{-\cos n\pi x}{n^2 \pi^2} \right) + (2) \left(\frac{\sin n\pi x}{n^3 \pi^3} \right) \right]_0^1$$

$$= 2 \left[\left(\frac{1}{n^2 \pi^2} \right) (2x - 2) \cos n\pi x \right]_0^1$$

$$= \frac{2}{n^2 \pi^2} [(0) - (-2 \cos 0)]$$

$$a_n = \frac{4}{n^2 \pi^2}$$

$$(1) \Rightarrow f(x) = \frac{2/3}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} \cos n\pi x$$

$$\therefore \boxed{f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi x} \text{----- (2)}$$

Deduction:

Let $x=0$ be a point of discontinuity

$$(2) \Rightarrow f(0) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos 0$$

$$\Rightarrow \frac{1}{2} = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \because f(x) = (x-1)^2 \Rightarrow f(0) = \frac{f(0) + f(1)}{2} = \frac{1+0}{2} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} - \frac{1}{3} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{1}{6} \times \frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{24}}$$

13. Find the Fourier cosine series for $x(\pi - x)$ in $0 < x < \pi$. Hence show that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$.

Solution:

Given:

$$\text{Let } f(x) = x(\pi - x), \quad 0 < x < \pi$$

$$f(x) = \pi x - x^2$$

Here $\ell = \text{Upper Limit} - \text{Lower Limit} = \pi - 0 = \pi \quad \therefore \ell = \pi$

Let the cosine series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \because l = \pi \text{ ----- (1)}$$

To Find a_0 :

$$a_0 = \frac{2}{\ell} \int_0^{\ell} f(x) dx = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) dx = \frac{2}{\pi} \left[\frac{\pi x^2}{2} - \frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \left[\left(\frac{\pi^3}{2} - \frac{\pi^3}{3} \right) \right] \text{ (0)}$$

$$= \frac{2}{\pi} \left[\frac{3\pi^3 - 2\pi^3}{6} \right] = \frac{2}{\pi} \left[\frac{\pi^3}{6} \right] = \frac{\pi^2}{3}$$

$$a_0 = \frac{\pi^2}{3}$$

To Find a_n :

$$a_n = 2 \int_0^{\pi} (x^2 - 2x + 1) \cos n\pi x dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \cos nx dx$$

$$= \frac{2}{\pi} \left[(\pi x - x^2) \left(\frac{\sin nx}{n} \right) - (\pi - 2x) \left(\frac{-\cos nx}{n^2} \right) + (-2) \left(\frac{-\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{1}{n^2} \right) (\pi - 2x) \cos nx \right]_0^{\pi}$$

$$= \frac{2}{\pi n^2} [-\pi \cos n\pi - \pi \cos 0] = \frac{-2\pi}{\pi n^2} [(-1)^n + 1]$$

$$= \frac{-2}{n^2} [(-1)^n + 1]$$

$$a_n = \begin{cases} -\frac{4}{n^2}, & \text{if } n=2,4,6,\dots \\ 0, & \text{if } n=1,3,5,\dots \end{cases}$$

$$\therefore f(x) = \frac{\pi^2}{6} + \sum_{n=2,4,\dots}^{\infty} -\frac{4}{n^2} \cos nx.$$



Deduction:

Let the Parseval's identity for Fourier cosine series be

$$\frac{2}{l} \int_0^l [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi} \int_0^{\pi} [\pi x - x^2]^2 dx = \frac{\pi^4}{2} + \sum_{n=2,4,6,\dots}^{\infty} \left(\frac{-4}{n^2} \right)^2$$

$$\frac{2}{\pi} \int_0^{\pi} [\pi^2 x^2 - 2\pi x^3 + x^4] dx = \frac{\pi^4}{18} + 16 \sum_{n=2,4,6,\dots}^{\infty} \frac{1}{n^4}$$



$$\frac{2}{\pi} \left[\left(\frac{\pi^5}{3} - \frac{\pi^5}{2} + \frac{\pi^5}{5} \right) - (0) \right] = \frac{\pi^4}{18} + 16 \sum_{n=1}^{\infty} \frac{1}{(2n)^4} \quad \therefore \sum_{n=2,4,6,\dots}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{2n}$$

$$\frac{2\pi^5}{\pi} \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = \frac{\pi^4}{18} + \frac{16}{16} \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$2\pi^4 \left(\frac{10-15+6}{30} \right) - \frac{\pi^4}{18} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^4}{15} - \frac{\pi^4}{18} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{6\pi^4 - 5\pi^4}{90}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}}$$

Find the cosine series for $f(x) = x$ in $(0, \pi)$ and then using Parseval's theorem, show that

14.

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$$

Solution:

Given:

Let $f(x) = x, 0 < x < \pi$

Here $\ell = \text{Upper Limit} - \text{Lower Limit} = \pi - 0 = \pi \quad \therefore \ell = \pi$

Let the cosine series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi \quad \therefore \ell = \pi$$

To Find a_0 :

$$a_0 = \frac{2}{\ell} \int_0^{\ell} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \frac{2}{2\pi} [\pi^2 - 0] = \pi$$

$$\boxed{a_0 = \pi}$$

To Find a_n :

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$$

$$= \frac{2}{\pi} \left[(x) \left(\frac{\sin nx}{n} \right) - (1) \left(\frac{-\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{1}{n^2} \right) \cos nx \right]_0^{\pi}$$

$$= \frac{2}{\pi n^2} [\cos n\pi - \cos 0]$$

$$= \frac{2}{n^2 \pi} [(-1)^n - 1]$$



$$a_n = \begin{cases} \frac{-4}{n^2\pi}, & \text{if } n = 1, 3, 5, \dots \\ 0, & \text{if } n = 2, 4, 6, \dots \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5,\dots}^{\infty} \frac{-4}{\pi n^2} \cos nx$$

Deduction:

Let the Parseval's identity for Fourier cosine series be

$$\frac{2}{l} \int_0^l [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{\pi^2}{2} + \sum_{n=1,3,5,\dots}^{\infty} \left(\frac{-4}{n^2\pi} \right)^2$$

$$\frac{2}{\pi} \left(\frac{x^3}{3} \right)_0^{\pi} = \frac{\pi^2}{2} + \sum_{n=1,3,5,\dots}^{\infty} \frac{16}{\pi^2 n^4}$$

$$\frac{2}{3\pi} (\pi^3 - 0) = \frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^4}$$

$$\frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^2}{6} \times \frac{\pi^2}{16} = \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^4}$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \infty = \frac{\pi^4}{96}$$

15. Find the complex form of the Fourier series of $f(x) = \cos ax$ in $(-\pi, \pi)$, where a is not an integer.

Solution:

Given $f(x) = \cos ax$ in $(-\pi, \pi)$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} \quad \therefore \ell = \pi$

Let the complex form of the Fourier series be

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in x} \quad \because l = \pi$$

$$C_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\frac{in\pi x}{l}} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos ax e^{-inx} dx$$

$$\therefore \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx] \quad \text{Here } a = -in \text{ \& } b = a$$

$$= \frac{1}{2\pi} \left\{ \frac{e^{-inx}}{((in)^2 - a^2)} (-in \cos ax + a \sin ax) \right\}_{-\pi}^{\pi}$$



$$= \frac{1}{2\pi(a^2 - n^2)} \left[e^{-in\pi} (-in \cos a\pi + a \sin a\pi) - e^{in\pi} (-in \cos a\pi - a \sin a\pi) \right]$$

$$e^{in\pi} = \cos n\pi + i \sin n\pi = \cos n\pi = (-1)^n \quad \text{and} \quad e^{-in\pi} = \cos n\pi - i \sin n\pi = \cos n\pi = (-1)^n \quad \therefore \sin n\pi = 0$$

$$= \frac{1}{2\pi(a^2 - n^2)} \left[(-1)^n [-in \cos a\pi + a \sin a\pi + in \cos a\pi + a \sin a\pi] \right]$$

$$= \frac{1}{2\pi(a^2 - n^2)} \left[(-1)^n (2a \sin a\pi) \right]$$

$$C_n = \frac{(-1)^n a \sin a\pi}{\pi(a^2 - n^2)}$$

$$\therefore f(x) = \frac{a \sin a\pi}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n}{(a^2 - n^2)} e^{inx}$$

16. Find complex form of the Fourier series of the function $f(x) = e^{-x}, -1 < x < 1$

Solution:

Given $f(x) = e^{-x}, -1 < x < 1$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{1 - (-1)}{2} = \frac{2}{2} = 1 \quad \therefore \ell = 1$$

Let the complex form of the Fourier series be

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x} \quad \therefore l = 1$$

$$C_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\frac{in\pi x}{l}} dx = \frac{1}{2} \int_{-1}^1 e^{-x} e^{-in\pi x} dx$$

$$C_n = \frac{1}{2} \int_{-1}^1 e^{-x - in\pi x} dx$$

$$= \frac{1}{2} \int_{-1}^1 e^{-(1+in\pi)x} dx$$

$$= \frac{1}{2} \left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)} \right]_{-1}^1$$

$$= -\frac{1}{2(1+in\pi)} \left\{ e^{-(1+in\pi)} - e^{(1+in\pi)} \right\}$$

$$= -\frac{1}{2(1+in\pi)} \left\{ e^{-1} e^{-in\pi} - e^1 e^{in\pi} \right\}$$

$$e^{in\pi} = \cos n\pi + i \sin n\pi = \cos n\pi = (-1)^n \quad \text{and} \quad e^{-in\pi} = \cos n\pi - i \sin n\pi = \cos n\pi = (-1)^n \quad \therefore \sin n\pi = 0$$

$$= \frac{-(1-in\pi)}{2(1+n^2\pi^2)} \left\{ e^{-1} (-1)^n - e^1 (-1)^n \right\}$$



$$= \frac{(1 - in\pi)}{2(1 + n^2\pi^2)} (-1)^n (e^1 - e^{-1})$$

$$C_n = \frac{(1 - in\pi)}{2(1 + n^2\pi^2)} (-1)^n 2\sinh(1) \quad \because \sinh \theta = \frac{e^\theta - e^{-\theta}}{2}$$

$$\therefore f(x) = \sum_{-\infty}^{\infty} \frac{(1 - in\pi)}{(1 + n^2\pi^2)} (-1)^n \sinh(1) e^{in\pi x}$$

17. Calculate the first two harmonic of the Fourier series of $f(x)$ from the following data

x	0	30	60	90	120	150	180	210	240	270	300	330
f(x)	1.8	1.1	0.3	0.16	0.5	1.3	2.16	1.25	1.3	1.52	1.76	2.0

Solution:

x	0	30	60	90	120	150	180	210	240	270	300	330	360
f(x)	1.8	1.1	0.3	0.16	0.5	1.3	2.16	1.25	1.3	1.52	1.76	2.0	1.8

We know that $360 = 2\pi$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \pi \quad \therefore \ell = \pi$$

$K=12$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \because \ell = \pi$$

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots$$

Where

$$a_0 = \frac{2}{K} \sum y \quad b_1 = \frac{2}{K} \sum y \sin x$$

$$a_1 = \frac{2}{K} \sum y \cos x \quad b_2 = \frac{2}{K} \sum y \sin 2x$$

$$a_2 = \frac{2}{K} \sum y \cos 2x$$

x	y	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
0	1.8	1.8	1.8	0	0
30	1.1	0.95	0.55	0.55	0.95
60	0.3	0.15	-0.15	0.26	0.26
90	0.16	0.00	-0.16	0.16	0.00
120	0.5	-0.25	-0.25	0.43	-0.43
150	1.3	-1.13	0.65	0.65	-1.12
180	2.16	-2.16	2.16	0.00	0.01
210	1.25	-1.08	0.62	-0.63	1.08
240	1.3	-0.65	-0.65	-1.13	1.12



270	1.52	0.00	-1.52	-1.52	-0.01
300	1.76	0.88	-0.87	-1.52	-1.53
330	2.0	1.73	1.01	-1.00	-1.73
Total	15.15	0.26	3.18	-3.74	-1.39

$$a_0 = \frac{2}{12}(15.15) = 2.52 \quad b_1 = \frac{2}{12}(-3.74) = -0.62$$

$$a_1 = \frac{2}{12}(0.26) = 0.043 \quad b_2 = \frac{2}{12}(-1.39) = -0.23$$

$$a_2 = \frac{2}{12}(3.18) = 0.53$$

$$f(x) = 1.26 + (0.043 \cos x - 0.62 \sin x) + (0.53 \cos 2x - 0.23 \sin 2x) + \dots$$

18. Find the first two harmonic of the Fourier series of f(x) given by

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
f(x)	1	1.4	1.9	1.7	1.5	1.2	1.0

Solution:

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
f(x)=y	1	1.4	1.9	1.7	1.5	1.2	1.0

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \pi \quad \therefore \ell = \pi$$

$$K=6$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \because \ell = \pi$$

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots$$

Where

$$a_0 = \frac{2}{K} \sum y \quad b_1 = \frac{2}{K} \sum y \sin x$$

$$a_1 = \frac{2}{K} \sum y \cos x \quad b_2 = \frac{2}{K} \sum y \sin 2x$$

$$a_2 = \frac{2}{K} \sum y \cos 2x$$

x	y	y cos x	y cos 2x	y sin x	y sin 2x
0	1	1	1	0	0
$\frac{\pi}{3} = 60$	1.4	0.7	-0.7	1.212	1.212
$\frac{2\pi}{3} = 120$	1.9	-0.95	-0.95	1.65	-1.645
$\pi = 180$	1.7	-1.7	1.7	0	0
$\frac{4\pi}{3} = 240$	1.5	-0.75	-0.75	-1.299	1.299



$\frac{5\pi}{3} = 300$	1.2	0.6	-0.6	-1.039	-1.039
Total	8.7	-1.1	-0.3	0.5196	-0.1732

$$a_0 = \frac{2}{6}(8.7) = 2.9 \quad b_1 = \frac{2}{6}(0.5196) = 0.17$$

$$a_1 = \frac{2}{6}(-1.1) = -0.37 \quad b_2 = \frac{2}{6}(-0.1732) = -0.06$$

$$a_2 = \frac{2}{6}(-0.3) = -0.1$$

$$f(x) = 1.45 + (-0.37 \cos x + 0.17 \sin x) + (-0.1 \cos 2x - 0.06 \sin 2x) + \dots$$

19. Find the first three harmonic of the Fourier series of f(x) given by

X	0	1	2	3	4	5
f(x)	9	18	24	28	26	20

Solution:

X	0	1	2	3	4	5	6
f(x)	9	18	24	28	26	20	9

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{6 - 0}{2} = 3 \quad \therefore \ell = 3$$

$$K = 6$$

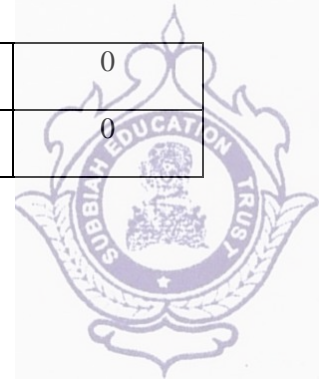
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3}$$

$$f(x) = \frac{a_0}{2} + \left(a_1 \cos \frac{\pi x}{3} + b_1 \sin \frac{\pi x}{3} \right) + \left(a_2 \cos \frac{2\pi x}{3} + b_2 \sin \frac{2\pi x}{3} \right) + \left(a_3 \cos \frac{3\pi x}{3} + b_3 \sin \frac{3\pi x}{3} \right) + \dots$$

Where

$a_0 = \frac{2}{K} \sum y$		$b_1 = \frac{2}{K} \sum y \sin \left(\frac{\pi x}{3} \right)$					
$a_1 = \frac{2}{K} \sum y \cos \left(\frac{\pi x}{3} \right)$		$b_2 = \frac{2}{K} \sum y \sin \left(\frac{2\pi x}{3} \right)$					
$a_2 = \frac{2}{K} \sum y \cos \left(\frac{2\pi x}{3} \right)$		$b_3 = \frac{2}{K} \sum y \sin \left(\frac{3\pi x}{3} \right)$					
$a_3 = \frac{2}{K} \sum y \cos \left(\frac{3\pi x}{3} \right)$							
<i>x</i>	<i>y</i>	$y \cos \frac{\pi x}{3}$ (or) $y \cos 60x$	$y \cos \frac{2\pi x}{3}$ $y \cos 120x$	$y \cos \frac{3\pi x}{3}$ $y \cos 180x$	$y \sin \frac{\pi x}{3}$ $y \sin 60x$	$y \sin \frac{2\pi x}{3}$ $y \sin 120x$	$y \sin \frac{3\pi x}{3}$ $y \cos 180x$
0	9	9	9	9	0	0	0
1	18	9	-9	-18	15.7	15.6	0
2	24	-12	-24	24	20.9	-20.784	0
3	28	-28	28	-28	0	0	0
4	26	-13	-13	26	-22.6	22.6	0



5	20	10	-10	-20	-17.4	-17.4	0
Total	125	-25	-19	-7	-3.4	20.8	0

$$a_0 = \frac{2}{6}(125) = 41.66 \quad b_1 = \frac{2}{6}(-3.4) = -1.13$$

$$a_1 = \frac{2}{6}(-25) = -8.33 \quad b_2 = \frac{2}{6}(20.8) = 6.9$$

$$a_2 = \frac{2}{6}(-19) = -6.33 \quad b_3 = \frac{2}{6}(0) = 0$$

$$a_3 = \frac{2}{6}(-7) = -2.3$$

The required Harmonic of the Fourier series be

$$f(x) = \frac{41.66}{2} + \left(-8.33 \cos \frac{\pi x}{3} + -1.13 \sin \frac{\pi x}{3}\right) + \left(-6.33 \cos \frac{2\pi x}{3} + 6.9 \sin \frac{2\pi x}{3}\right) + \left(-2.3 \cos \frac{3\pi x}{3} + 0\right) + \dots$$

$$f(x) = 20.83 + \left(-8.33 \cos \frac{\pi x}{3} + -1.13 \sin \frac{\pi x}{3}\right) + \left(-6.33 \cos \frac{2\pi x}{3} + 6.9 \sin \frac{2\pi x}{3}\right) + \left(-2.3 \cos \frac{3\pi x}{3}\right) + \dots$$

20. The following table gives the variations of a periodic function over a period T

x	0	T/6	T/3	T/2	2T/3	5T/6	T
f(x)	1.98	1.3	1.05	1.3	-0.88	-0.25	1.98

find f(x) upto first harmonic.

Solution:

Assume $X = \frac{2\pi x}{T}$

X	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{2\pi}{3}$
y=f(θ)	1.98	1.3	1.05	1.3	-0.88	-0.25	1.98

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \pi \quad \therefore \ell = \pi$$

$$K=6$$

$$f(X) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi X}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi X}{\ell}$$

$$f(X) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nX + \sum_{n=1}^{\infty} b_n \sin nX$$

$$f(x) = \frac{a_0}{2} + (a_1 \cos nX + b_1 \sin nX) + (a_2 \cos 2X + b_2 \sin 2X) + (a_3 \cos 3X + b_3 \sin 3X) + \dots$$

Where

$a_0 = \frac{2}{K} \sum y$			$a_1 = \frac{2}{K} \sum y \cos X$		$b_1 = \frac{2}{K} \sum y \sin X$		
x	y	y cos X	y sin X				
0	1.98	1.98	0				
$\frac{\pi}{3}$	1.30	0.65	1.1258				

$\frac{2\pi}{3}$	1.05	-0.525	0.9093
π	1.30	-1.3	0
$\frac{4\pi}{3}$	-0.88	0.44	0.762
$\frac{5\pi}{3}$	-0.25	-0.125	0.2165
Total	4.6	1.12	3.013

$$a_0 = \frac{2}{6} \sum y = \frac{4.6}{3} = 1.5 \quad a_1 = \frac{2}{6}(1.12) = 0.37 \quad b_1 = \frac{2}{6}(3.013) = 1.005$$

$$\therefore \boxed{f(x) = 0.75 + 0.37 \cos X + 1.005 \sin X}$$





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UNIT II – FOURIER SERIES

2.3 Harmonic Analysis

The process of finding the Fourier series for a function given by numerical values is known as harmonic analysis.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \text{ where}$$

ie, $f(x) = (a_0/2) + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x) + \dots(1)$

$$\text{Here } a_0 = 2 [\text{mean values of } f(x)] = \frac{2 \sum f(x)}{n}$$

$$a_n = 2 [\text{mean values of } f(x) \cos nx] = \frac{2 \sum f(x) \cos nx}{n}$$

$$\& \quad b_n = 2 [\text{mean values of } f(x) \sin nx] = \frac{2 \sum f(x) \sin nx}{n}$$

In (1), the term $(a_1 \cos x + b_1 \sin x)$ is called the **fundamental or first harmonic**, the term $(a_2 \cos 2x + b_2 \sin 2x)$ is called the **second harmonic** and so on.

Problem 1.

Compute the first three harmonics of the Fourier series of $f(x)$ given by the following table.

x:	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
f(x):	1.0	1.4	1.9	1.7	1.5	1.2	1.0



We exclude the last point $x = 2\pi$.

Let $f(x) = (a_0/2) + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots$

To evaluate the coefficients, we form the following table.

x	f(x)	cosx	sinx	cos2x	sin2x	cos3x	sin3x
0	1.0	1	0	1	0	1	0
$\pi/3$	1.4	0.5	0.866	-0.5	0.866	-1	0
$2\pi/3$	1.9	-0.5	0.866	-0.5	-0.866	1	0
π	1.7	-1	0	1	0	-1	0
$4\pi/3$	1.5	-0.5	-0.866	-0.5	0.866	1	0
$5\pi/3$	1.2	0.5	-0.866	-0.5	-0.866	-1	0

$$\text{Now, } a_0 = \frac{2 \sum f(x)}{6} = \frac{2(1.0 + 1.4 + 1.9 + 1.7 + 1.5 + 1.2)}{6} = 2.9$$

$$a_1 = \frac{2 \sum f(x) \cos x}{6} = -0.37$$

$$a_2 = \frac{2 \sum f(x) \cos 2x}{6} = -0.1$$

$$a_3 = \frac{2 \sum f(x) \cos 3x}{6} = 0.033$$

$$b_1 = \frac{2 \sum f(x) \sin x}{6} = 0.17$$

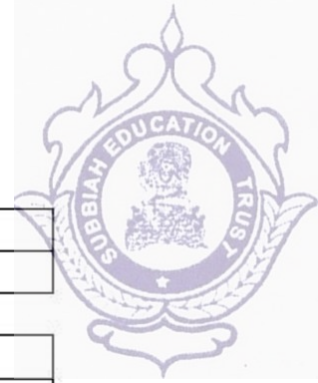
$$b_2 = \frac{2 \sum f(x) \sin 2x}{6} = -0.06$$

$$b_3 = \frac{2 \sum f(x) \sin 3x}{6} = 0$$

$$\therefore f(x) = 1.45 - 0.37 \cos x + 0.17 \sin x - 0.1 \cos 2x - 0.06 \sin 2x + 0.033 \cos 3x + \dots$$

Problem 2

Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table:



x:	0	1	2	3	4	5
y:	4	8	15	7	6	2

Taking the interval as 60° , we have

θ :	0°	60°	120°	180°	240°	300°
x:	0	1	2	3	4	5
y:	4	8	15	7	6	2

\therefore Fourier cosine series in the interval $(0, 2\pi)$ is $y = (a_0/2) + a_1\cos\theta + a_2\cos2\theta + a_3\cos3\theta + \dots$

To evaluate the coefficients, we form the following table.

θ°	$\cos\theta$	$\cos2\theta$	$\cos3\theta$	y	y cos θ	y cos 2θ	y cos 3θ
0°	1	1	1	4	4	4	4
60°	0.5	-0.5	-1	8	4	-4	-8
120°	-0.5	-0.5	1	15	-7.5	-7.5	15
180°	-1	1	-1	7	-7	7	-7
240°	-0.5	-0.5	1	6	-3	-3	6
300°	0.5	-0.5	-1	2	1	-1	-2
Total				42	-8.5	-4.5	8

Now, $a_0 = 2 (42/6) = 14$

$a_1 = 2 (-8.5/6) = -2.8$

$a_2 = 2 (-4.5/6) = -1.5$

$a_3 = 2 (8/6) = 2.7$

$y = 7 - 2.8 \cos\theta - 1.5 \cos2\theta + 2.7 \cos3\theta + \dots$

Problem 3

The values of x and the corresponding values of f(x) over a period T are given below. Show that $f(x) = 0.75 + 0.37 \cos\theta + 1.004 \sin\theta$, where $\theta = (2\pi x) / T$

x:	0	T/6	T/3	T/2	2T/3	5T/6	T
y:	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

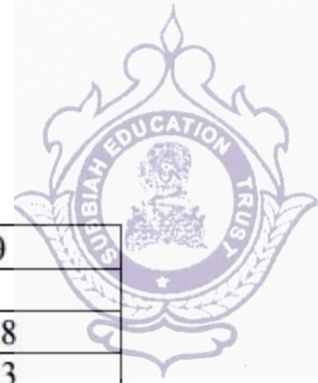
We omit the last value since f(x) at x = 0 is known.

Here $\theta = 2\pi x / T$

When x varies from 0 to T, θ varies from 0 to 2π with $2\pi/6$ as an increment.

Let $f(x) = F(\theta) = (a_0/2) + a_1 \cos\theta + b_1 \sin\theta$.

To evaluate the coefficients, we form the following table.



θ	y	$\cos\theta$	$\sin\theta$	y $\cos\theta$	y $\sin\theta$
0	1.98	1.0	0	1.98	0
$\pi/3$	1.30	0.5	0.866	0.65	1.1258
$2\pi/3$	1.05	-0.5	0.866	-0.525	0.9093
Π	1.30	-1	0	-1.3	0
$4\pi/3$	-0.88	-0.5	-0.866	0.44	0.762
$5\pi/3$	-0.25	0.5	-0.866	-0.125	0.2165
	4.6			1.12	3.013

$$\text{Now, } a_0 = 2 \left(\sum f(x)/6 \right) = 1.5$$

$$a_1 = 2 (1.12 / 6) = 0.37$$

$$a_2 = 2 (3.013/6) = 1.004$$

$$\text{Therefore, } f(x) = 0.75 + 0.37 \cos\theta + 1.004 \sin\theta$$



DEPARTMENT OF MATHEMATICS

UNIT II – FOURIER SERIES

2.4 PERIODIC FUNCTION

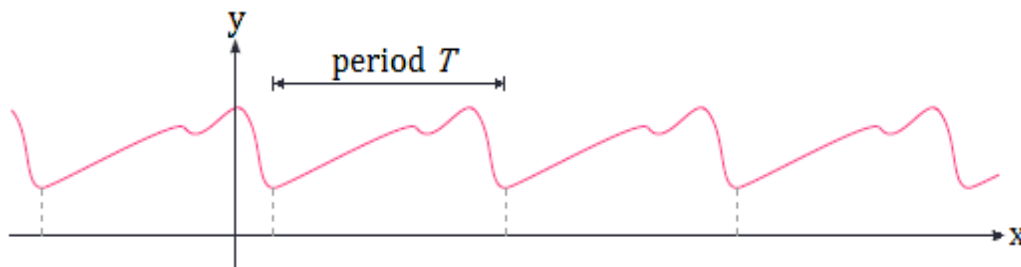
Periodic Functions

- A function $f(x)$ is periodic if there is a positive number T such that for every value of x :

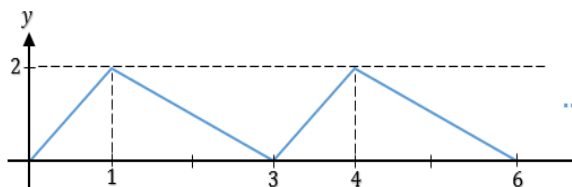
$$f(x + T) = f(x)$$

The smallest such value of $T > 0$ is called the fundamental period or simply the period.

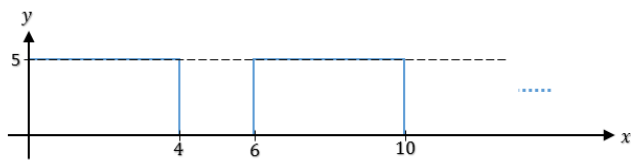
- If we know what the graph looks like in an interval of length T , then we can use replication to sketch the entire graph.



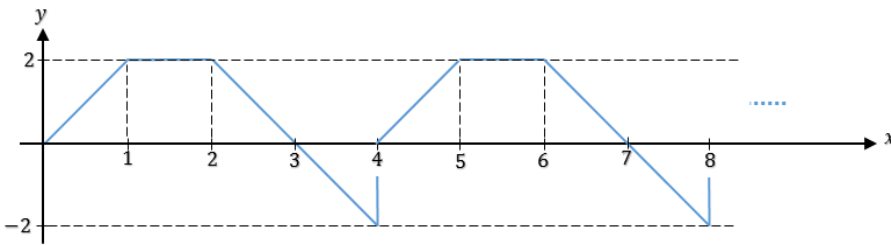
Problem 1: What is the fundamental period of the following function?



(a)



(b)



(c)

Solution:

- a) The period is 3
- b) The period is 2
- c) The period is 4

Period of Trigonometric Functions

Period π :

$$\tan(x + \pi) = \tan x$$

$$\cot(x + \pi) = \cot x$$

$$\tan x = \frac{\sin x}{\cos x} \quad , \quad \cot x = \frac{1}{\tan x}$$

$$\sec x = \frac{1}{\cos x} \quad , \quad \csc x = \frac{1}{\sin x}$$

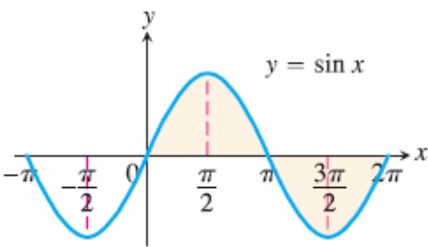
Period 2π :

$$\sin(x + 2\pi) = \sin x$$

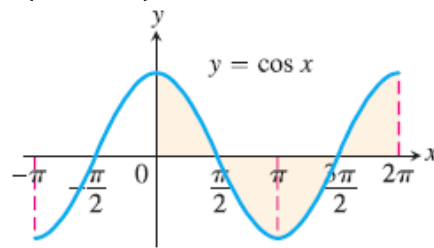
$$\cos(x + 2\pi) = \cos x$$

$$\sec(x + 2\pi) = \sec x$$

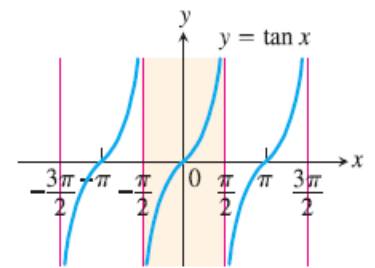
$$\csc(x + 2\pi) = \csc x$$



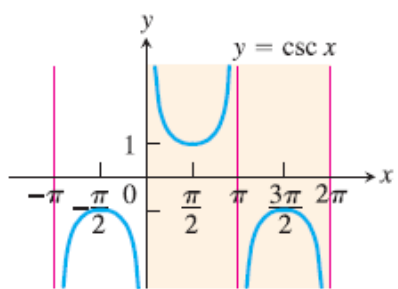
Domain: $-\infty < x < \infty$
 Range: $-1 \leq y \leq 1$
 Period: 2π



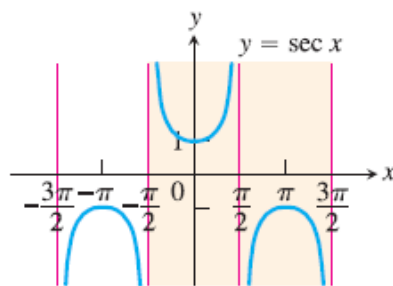
Domain: $-\infty < x < \infty$
 Range: $-1 \leq y \leq 1$
 Period: 2π



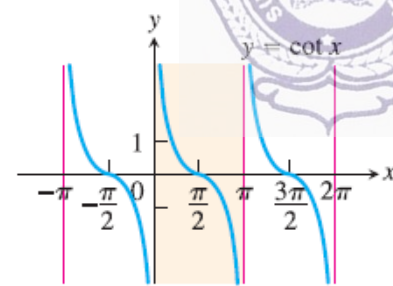
Domain: $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$
 Range: $-\infty < y < \infty$
 Period: π



Domain: $x \neq 0, \pm\pi, \pm2\pi, \dots$
 Range: $y \leq -1$ or $y \geq 1$
 Period: 2π



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$
 Range: $y \leq -1$ or $y \geq 1$
 Period: 2π

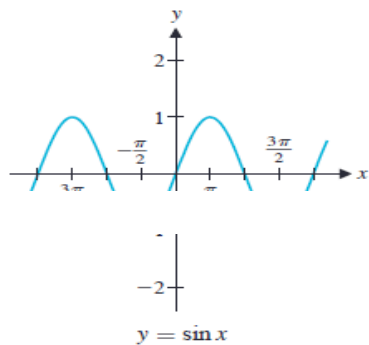


Domain: $x \neq 0, \pm\pi, \pm2\pi, \dots$
 Range: $-\infty < y < \infty$
 Period: π

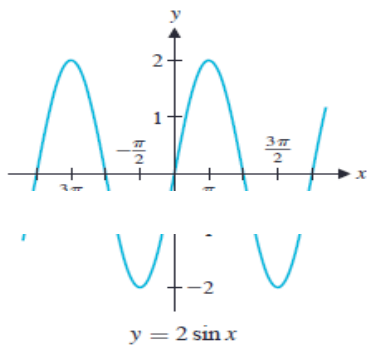
Problem 2: Sketch a) $y = \sin x$, b) $y = 2 \sin x$, and c) $y = \sin(2x)$,
 What is the period of each function?

Solution:

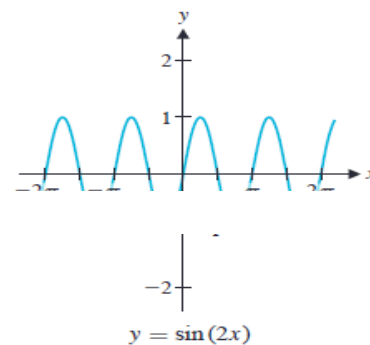
a) The period is 2π



b) The period is 2π



c) The period is π





DEPARTMENT OF MATHEMATICS

UNIT II – FOURIER SERIES

2.5 EVEN & ODD FUNCTION - HALF RANGE FOURIER SERIES

A function $y = f(x)$ is an:

Even function of x if $f(-x) = f(x)$,

Odd function of x if $f(-x) = -f(x)$,

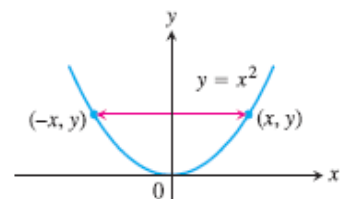
for every x in the function's domain.

The graphs of even and odd functions have characteristic symmetry properties.

- The graph of an **even function** is **symmetric about the y-axis**.
- The graph of an **odd function** is **symmetric about the origin**.

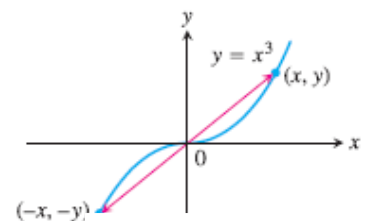
Problem 1:

$f(x) = x^2$ is an even function, since $(-x)^2 = x^2$ for all x , (symmetry about y-axis).



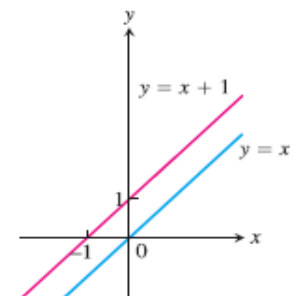
Problem 2:

$f(x) = x^3$ is an odd function, since $(-x)^3 = -x^3$ for all x , (symmetry about the origin).



Problem 3:

$f(x) = x$ is an odd function, since $(-x) = -x$ for all x , (symmetry about the origin).



**Problem 4:**

$f(x) = x + 1$ is neither even nor odd:

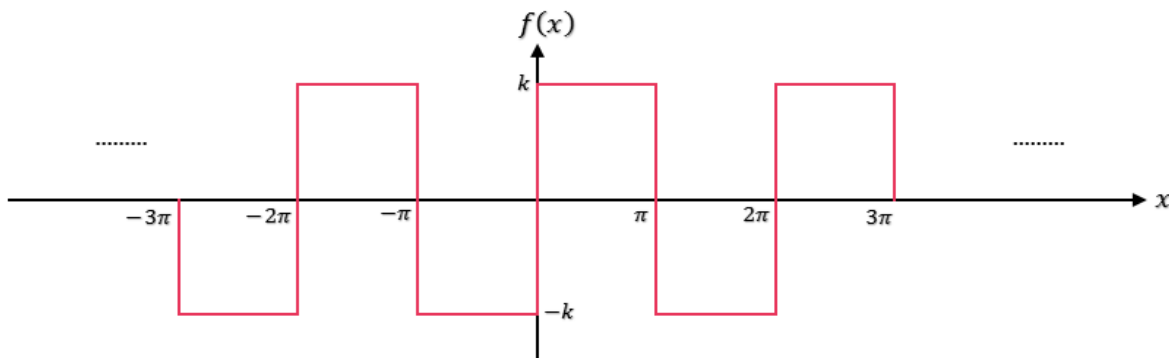
Not even function since, $f(-x) \neq f(x)$ for all $x \neq 0$ where $f(-x) = (-x) + 1$, but $f(x) = x + 1$.

Not odd function since, $f(-x) \neq -f(x)$ where $f(-x) = -x + 1$, but $-f(x) = -x - 1$.

Problem 5: Find the Fourier series corresponding to the following periodic function:

$$f(x) = \begin{cases} -k, & \text{for } -\pi < x \leq 0 \\ k, & \text{for } 0 < x \leq \pi \end{cases}$$

where $f(x) = f(x + 2\pi)$

**Solution:**

The period $T = 2L = 2\pi \Rightarrow L = \pi$

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^L f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 (-k) dx + \frac{1}{\pi} \int_0^{\pi} (k) dx \\ &= \frac{-k}{\pi} x \Big|_{-\pi}^0 + \frac{k}{\pi} x \Big|_0^{\pi} \end{aligned}$$



$$= \frac{-k}{\pi}(0 + \pi) + \frac{k}{\pi}(\pi - 0) = -k + k = 0$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos \frac{n\pi x}{\pi} dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 (-k) \cos nx dx + \frac{1}{\pi} \int_0^{\pi} (k) \cos nx dx \\ &= \frac{-k}{n\pi} \sin nx \Big|_{-\pi}^0 + \frac{k}{n\pi} \sin nx \Big|_0^{\pi} \\ &= \frac{-k}{n\pi} [\sin 0 - \sin(-n\pi)] + \frac{k}{n\pi} [\sin \pi n - \sin 0] = 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin \frac{n\pi x}{\pi} dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 (-k) \sin nx dx + \frac{1}{\pi} \int_0^{\pi} (k) \sin nx dx \\ &= \frac{k}{n\pi} \cos nx \Big|_{-\pi}^0 - \frac{k}{n\pi} \cos nx \Big|_0^{\pi} \\ &= \frac{k}{n\pi} [\cos 0 - \cos(-n\pi)] - \frac{k}{n\pi} (\cos n\pi - \cos 0) \\ &= \frac{k}{n\pi} [1 - \cos(n\pi)] - \frac{k}{n\pi} (\cos n\pi - 1) \\ &= \frac{2k}{n\pi} [1 - \cos(n\pi)] \\ &= \frac{2k}{n\pi} (1 - (-1)^n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4k}{n\pi}, & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$



we can write the even- indexed and odd-indexed coefficients separately as

$$b_{(2n)} = 0, \text{ for } k = 1, 2, \dots \text{ and } b_{(2n-1)} = \frac{4k}{(2n-1)\pi}, \text{ for } k = 1, 2, \dots$$

$$\begin{aligned} \therefore f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] \\ &= \sum_{n=1}^{\infty} b_{(2n-1)} \sin \frac{(2n-1)\pi x}{\pi} \\ &= \sum_{n=1}^{\infty} \frac{4k}{(2n-1)\pi} \sin(2n-1)x \\ &= \frac{4k}{\pi} \sin x + \frac{4k}{3\pi} \sin 3x + \frac{4k}{5\pi} \sin 5x + \dots \end{aligned}$$

Half Range Fourier series

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval $(0, L)$, which is half of the interval $(-L, L)$, and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, $(-L, 0)$.



For half range Fourier sine series (odd functions):

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad \text{for } 0 \leq x \leq L$$

For half range Fourier cosine series (even functions):

$$a_0 = \frac{2}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = 0$$

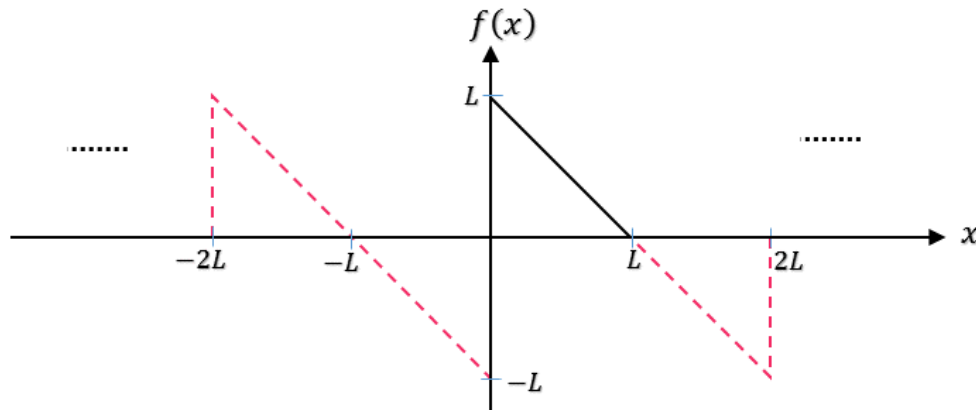
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad \text{for } 0 \leq x \leq L$$



Problem 6: Determine the half range Fourier sine series corresponding to:
 $f(x) = L - x$, for $0 < x < L$.

Solution:

A Fourier series consisting of sine terms alone is obtained only for an odd function. Hence, we extend the definition of $f(x)$ so that it becomes odd.



Taking the period $T = 2L$

Since the function is odd then: $a_0 = 0$ and $a_n = 0$

$$\begin{aligned}
 b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L (L - x) \left(\sin \frac{n\pi x}{L} \right) dx \\
 &= \frac{2}{L} \left[(L - x) \left(\frac{-\cos \left(\frac{n\pi x}{L} \right)}{\left(\frac{n\pi}{L} \right)} \right) - (-1) \left(\frac{-\sin \left(\frac{n\pi x}{L} \right)}{\left(\frac{n\pi}{L} \right)^2} \right) \right]_0^L \\
 &= \frac{2}{L} \left[\frac{L}{n\pi} (x - L) \left(\cos \left(\frac{n\pi x}{L} \right) \right) - \frac{L^2}{n^2 \pi^2} \left(\sin \left(\frac{n\pi x}{L} \right) \right) \right]_0^L \\
 &= \frac{2}{L} \left(\frac{L}{n^2 \pi^2} \right) \left[n\pi (x - L) \left(\cos \left(\frac{n\pi x}{L} \right) \right) - L \sin \left(\frac{n\pi x}{L} \right) \right]_0^L \\
 &= \left(\frac{2}{n^2 \pi^2} \right) \left[0 - n\pi (0 - L) (\cos 0) - L \sin \left(\frac{n\pi L}{L} \right) + L \sin 0 \right] \\
 &= \left(\frac{2}{n^2 \pi^2} \right) [n\pi L] = \frac{2L}{n\pi}
 \end{aligned}$$

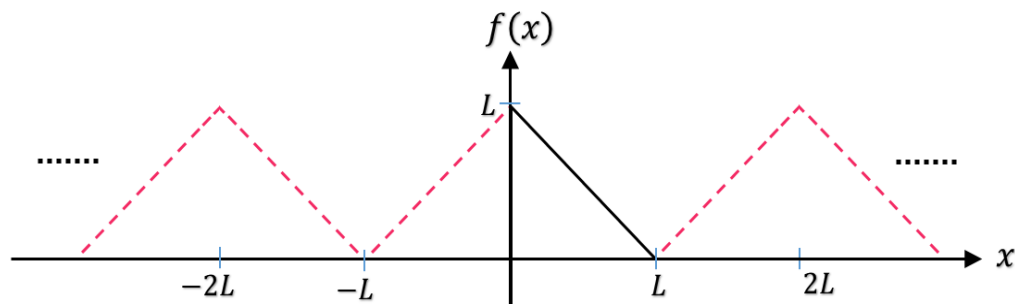


$$\begin{aligned} \therefore f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad \text{for } 0 \leq x \leq L \\ &= \sum_{n=1}^{\infty} \left(\frac{2L}{n\pi} \right) \sin \frac{n\pi x}{L} \\ &= \left(\frac{2L}{\pi} \right) \sin \frac{\pi x}{L} + \left(\frac{2L}{2\pi} \right) \sin \frac{2\pi x}{L} + \left(\frac{2L}{3\pi} \right) \sin \frac{3\pi x}{L} + \dots \end{aligned}$$

Problem 7: Determine the half range Fourier cosine series corresponding to:
 $f(x) = L - x$, for $0 < x < L$.

Solution:

A Fourier series consisting of cosine terms alone is obtained only for an even function. Hence, we extend the definition of $f(x)$ so that it becomes even.



Taking the period $T = 2L$

Since the function is even then: $b_0 = 0$

$$\begin{aligned} a_0 &= \frac{2}{L} \int_0^L f(x) dx = \frac{2}{L} \int_0^L (L - x) dx = \frac{2}{L} \left[Lx - \frac{x^2}{2} \right]_0^L = \frac{2}{L} \left(L^2 - \frac{L^2}{2} \right) = \frac{2}{L} \left(\frac{L^2}{2} \right) \\ &= L \\ a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L (L - x) \cos \frac{n\pi x}{L} dx \end{aligned}$$



$$\begin{aligned}
 &= \frac{2}{L} \left[(L-x) \left(\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)} \right) - (-1) \left(\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2} \right) \right]_0^L \\
 &= \frac{2}{L} \left[\frac{L}{n\pi} (L-x) \left(\sin\left(\frac{n\pi x}{L}\right) \right) - \frac{L^2}{n^2\pi^2} \left(\cos\left(\frac{n\pi x}{L}\right) \right) \right]_0^L \\
 &= \frac{2}{L} \left(\frac{L}{n^2\pi^2} \right) [0 - n\pi(L-0) \sin(0) - L \cos(n\pi) + L \cos(0)] \\
 &= \left(\frac{2}{n^2\pi^2} \right) [-L \cos(n\pi) + L]
 \end{aligned}$$

$$= \left(\frac{2L}{n^2\pi^2} \right) [1 - (-1)^n] = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4L}{n^2\pi^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad \text{for } 0 \leq x \leq L$$

$$= \frac{L}{2} + \sum_{n=1}^{\infty} a_{(2n-1)} \cos \frac{(2n-1)\pi x}{L}$$

$$= \frac{L}{2} + \sum_{n=1}^{\infty} \left(\frac{4L}{(2n-1)^2\pi^2} \right) \cos \frac{(2n-1)\pi x}{L}$$

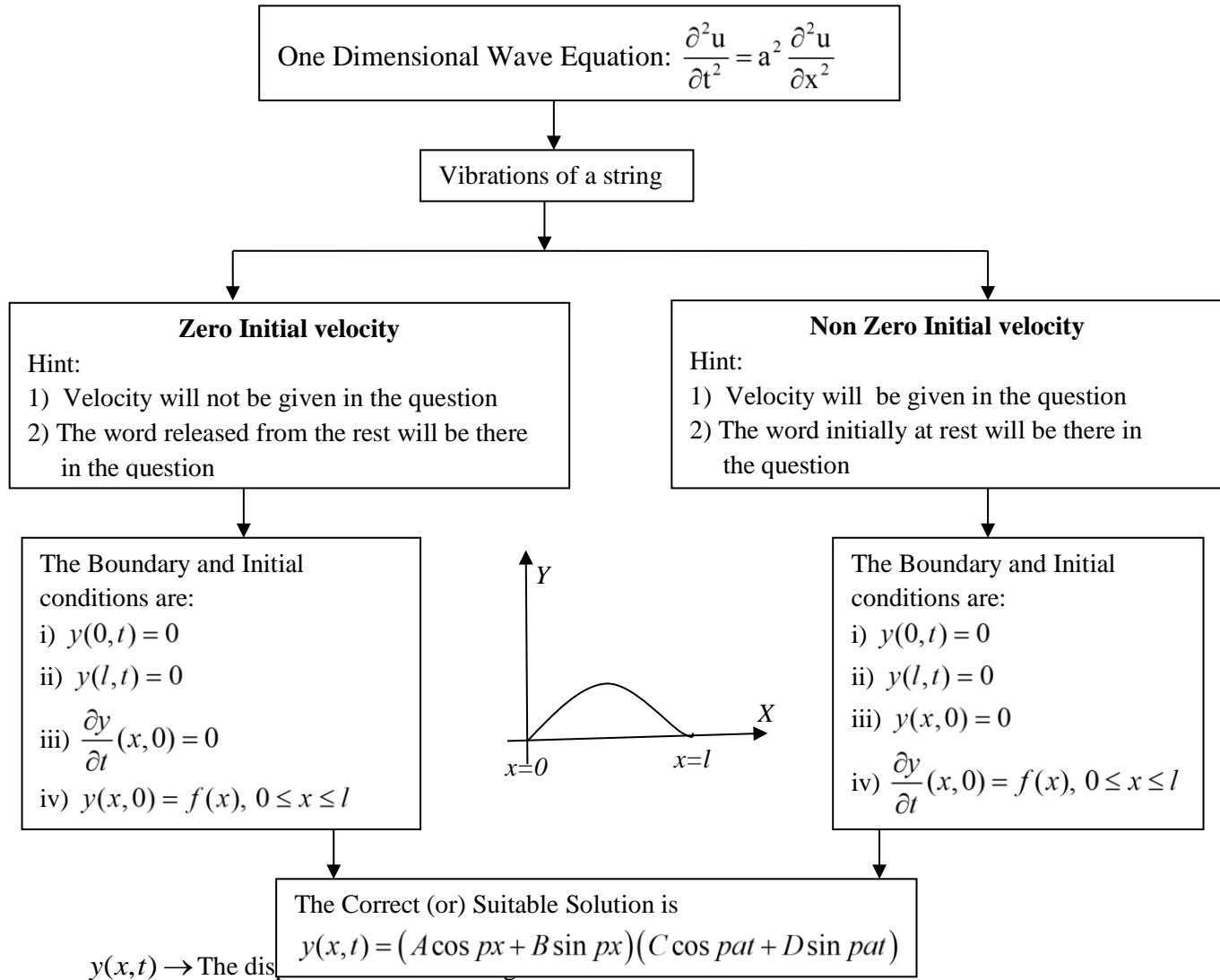
$$= \frac{L}{2} + \left(\frac{4L}{(1)^2\pi^2} \right) \cos \frac{\pi x}{L} + \left(\frac{4L}{(3)^2\pi^2} \right) \cos \frac{3\pi x}{L} + \left(\frac{4L}{(5)^2\pi^2} \right) \cos \frac{5\pi x}{L} + \dots$$

$$= \frac{L}{2} + \left(\frac{4L}{\pi^2} \right) \cos \frac{\pi x}{L} + \left(\frac{4L}{9\pi^2} \right) \cos \frac{3\pi x}{L} + \left(\frac{4L}{25\pi^2} \right) \cos \frac{5\pi x}{L} + \dots$$



UNIT III APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

3.1 ONE DIMENSIONAL WAVE EQUATION



1) In the wave equation $\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2}$, what does C^2 stands for?

Solution:

One dimensional heat equation is $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$

$C^2 = T/m$, where T is the tension and m is the mass of the string.

2) Write all possible solutions of the transverse vibration of the string in one dimension.

Solution:



(i) $y(x, t) = (Ae^{px} + Be^{-px}) (Ce^{pat} + De^{-pat})$

(ii) $y(x, t) = (A \cos px + B \sin px) (C \cos pat + D \sin pat)$.

(iii) $y(x, t) = (Ax + B) (Ct + D)$

PART - B

1. **A uniform string is stretched and fastened to two points 'l' apart. Motion is started by displacing the string into the form of the curve $y = kx(l - x)$ and then releasing it from this position at time $t = 0$. Find the displacement of the point of the string at a distance x from one end at time t .**

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x, t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \text{ ----- (1)}$$

The Boundary and Initial conditions are

i) $y(0, t) = 0$

ii) $y(l, t) = 0$

iii) $\frac{\partial y}{\partial t}(x, 0) = 0$

iv) $y(x, 0) = f(x) = kx(l - x), \quad 0 < x < l$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0, t) = (A \cos 0 + \cancel{B \sin 0})(C \cos pat + D \sin pat)$$

$$0 = (A)(C \cos pat + D \sin pat)$$

Here $(C \cos pat + D \sin pat) \neq 0 \therefore \boxed{A = 0}$

Sub $A=0$ in (1)

$$(1) \Rightarrow y(x, t) = (B \sin px)(C \cos pat + D \sin pat) \text{ ----- (2)}$$



Applying condn (ii) in (2)

$$y(l,t) = (B \sin pl)(C \cos pat + D \sin pat)$$

$$0 = (B \sin pl)(C \cos pat + D \sin pat)$$

Here $B \neq 0$, $(C \cos pat + D \sin pat) \neq 0$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + D \sin \frac{n\pi a}{l} t \right) \text{ ----- (3)}$$

Diff (3) partially w.r.to 't'

$$(2) \Rightarrow \frac{\partial y}{\partial t}(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left[-C \sin \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right) + D \cos \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right) \right]$$

Apply condn. (iii) in the above equation

$$\frac{\partial y}{\partial t}(x,0) = \left(B \sin \frac{n\pi}{l} x \right) \left[-C \sin 0 \times \left(\frac{n\pi a}{l} \right) + D \cos 0 \times \left(\frac{n\pi a}{l} \right) \right]$$

$$0 = \left(B \sin \frac{n\pi}{l} x \right) \left[D \times \left(\frac{n\pi a}{l} \right) \right]$$

Here $B \neq 0$, $\sin \frac{n\pi}{l} x \neq 0$, $\frac{n\pi a}{l} \neq 0$, $\therefore \boxed{D=0}$

Sub the value of D in (3)

$$(3) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + 0 \right)$$

$$y(x,t) = BC \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t$$



$$y(x,t) = b_1 \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \quad \text{let } BD = b_1$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \quad \text{----- (4)}$$

Applying condn (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \quad \because \cos 0 = 1$$

Which is half range Fourier sine series in $(0,l)$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l kx(l-x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2k}{l} \left[\cancel{(lx-x^2)} \left(\frac{-\cancel{\cos \frac{n\pi x}{l}}}{\frac{n\pi}{l}} \right) - (l-2x) \left(\frac{-\cancel{\sin \frac{n\pi x}{l}}}{\frac{n^2 \pi^2}{l^2}} \right) + (-2) \left(\frac{\cos \frac{n\pi x}{l}}{\frac{n^3 \pi^3}{l^3}} \right) \right]_0^l$$

$$= \frac{2k}{l} \left[\frac{-2l^3}{n^3 \pi^3} \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-4kl^3}{ln^3 \pi^3} [\cos n\pi - \cos 0]$$

$$= \frac{-4kl^2}{n^3 \pi^3} [(-1)^n - 1]$$



$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8kl^2}{n^3\pi^3} & \text{if } n \text{ is odd} \end{cases}$$

Sub b_n in (4)

$$y(x,t) = \sum_{n=1,3,5,\dots}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t$$

$$\text{(or)} \quad y(x,t) = \frac{8kl^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{(2n-1)\pi}{l} x \cos \frac{(2n-1)\pi a}{l} t$$

2. A string of length $2l$ is fastened at both ends. The midpoint of the string is displaced transversely through a small distance 'b' and the string is released from the rest in that position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \text{ ----- (1)}$$

The Boundary and Initial conditions are

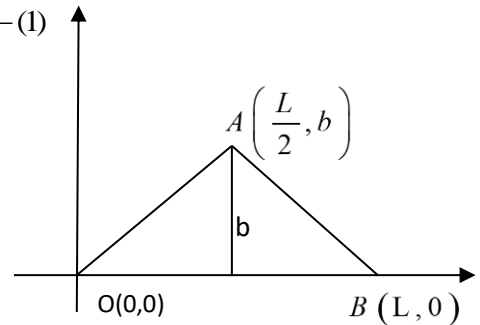
Assume $2l=L$

i) $y(0,t) = 0$

ii) $y(L,t) = 0$

iii) $\frac{\partial y}{\partial t}(x,0) = 0$

iv) $y(x,0) = f(x) = ?$



To find $f(x)$:

The equation of line joining two points is



$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$$

Equation of OA is $O(0,0)$ & $A(L/2,b)$:

$$\frac{x-0}{\frac{L}{2}-0} = \frac{y-0}{b-0} \Rightarrow \frac{2x}{L} = \frac{y}{b} \Rightarrow y = \frac{2b}{L}x, \quad 0 < x < \frac{L}{2}$$

Equation of AB is $A(L/2,b)$ & $B(L,0)$:

$$\frac{x-\frac{L}{2}}{L-\frac{L}{2}} = \frac{y-b}{0-b} \Rightarrow \frac{2x-L}{\frac{L}{2}} = \frac{y-b}{-b} \Rightarrow \frac{2x-L}{L} = \frac{y-b}{-b}$$

$$-2xb + lb = yl - lb$$

$$-2xb + Lb + Lb = yL \Rightarrow -2xb + 2Lb = yL$$

$$y = \frac{2b}{L}(L-x), \quad \frac{L}{2} < x < L$$

$$y = f(x) = \begin{cases} \frac{2b}{L}x, & 0 < x < \frac{L}{2} \\ \frac{2b}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0,t) = (A \cos 0 + \cancel{B \sin 0})(C \cos pat + D \sin pat)$$

$$0 = (A)(C \cos pat + D \sin pat)$$

Here $(C \cos pat + D \sin pat) \neq 0 \therefore \boxed{A=0}$

Sub $A=0$ in (1)

$$(1) \Rightarrow y(x,t) = (B \sin px)(C \cos pat + D \sin pat) \text{ ----- (2)}$$

Applying condn (ii) in (2)



$$y(L, t) = (B \sin pL)(C \cos pat + D \sin pat)$$

$$0 = (B \sin pL)(C \cos pat + D \sin pat)$$

Here $B \neq 0$, $(C \cos pat + D \sin pat) \neq 0$

$$\therefore \sin pL = 0 \Rightarrow \sin pL = \sin n\pi \Rightarrow pL = n\pi \Rightarrow p = \frac{n\pi}{L}$$

Sub the value of p in (2)

$$(2) \Rightarrow y(x, t) = \left(B \sin \frac{n\pi}{L} x \right) \left(C \cos \frac{n\pi a}{L} t + D \sin \frac{n\pi a}{L} t \right) \text{ ----- (3)}$$

Diff (3) partially w.r.to 't'

$$(2) \Rightarrow \frac{\partial y}{\partial t}(x, t) = \left(B \sin \frac{n\pi}{L} x \right) \left[-C \sin \frac{n\pi a}{L} t \times \left(\frac{n\pi a}{L} \right) + D \cos \frac{n\pi a}{L} t \times \left(\frac{n\pi a}{L} \right) \right]$$

Apply condn. (iii) in the above equation

$$\frac{\partial y}{\partial t}(x, 0) = \left(B \sin \frac{n\pi}{L} x \right) \left[-C \sin 0 \times \left(\frac{n\pi a}{L} \right) + D \cos 0 \times \left(\frac{n\pi a}{L} \right) \right]$$

$$0 = \left(B \sin \frac{n\pi}{L} x \right) \left[D \times \left(\frac{n\pi a}{L} \right) \right]$$

Here $B \neq 0$, $\sin \frac{n\pi}{L} x \neq 0$, $\frac{n\pi a}{L} \neq 0$, $\therefore D = 0$

Sub the value of D in (3)

$$(3) \Rightarrow y(x, t) = \left(B \sin \frac{n\pi}{L} x \right) \left(C \cos \frac{n\pi a}{L} t + 0 \right)$$

$$y(x, t) = BC \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t$$

$$y(x, t) = b_1 \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t \quad \text{let } BD = b_1$$



The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t \quad \text{----- (4)}$$

Applying condn (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad \because \cos 0 = 1$$

Which is half range Fourier sine series in $(0,l)$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$= \frac{2}{L} \left[\int_0^{\frac{L}{2}} \frac{2b}{L} x \sin \frac{n\pi x}{L} dx + \int_{\frac{L}{2}}^L \frac{2b}{L} (L-x) \sin \frac{n\pi x}{L} dx \right] \quad \because y = f(x) = \begin{cases} \frac{2b}{L} x, & 0 < x < \frac{L}{2} \\ \frac{2b}{L} (L-x), & \frac{L}{2} < x < L \end{cases}$$

$$= \frac{4b}{L^2} \left[\int_0^{\frac{L}{2}} x \sin \frac{n\pi x}{L} dx + \int_{\frac{L}{2}}^L (L-x) \sin \frac{n\pi x}{L} dx \right]$$

$$= \frac{4b}{L^2} \left\{ \left[(x) \left(\frac{-\cos \frac{n\pi x}{L}}{\frac{n\pi}{L}} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{L}}{\frac{n^2 \pi^2}{L^2}} \right) \right]_0^{\frac{L}{2}} + \left[(L-x) \left(\frac{-\cos \frac{n\pi x}{L}}{\frac{n\pi}{L}} \right) - (-1) \left(\frac{-\sin \frac{n\pi x}{L}}{\frac{n^2 \pi^2}{L^2}} \right) \right]_{\frac{L}{2}}^L \right\}$$

$$= \frac{4b}{L^2} \left\{ \left[-\frac{L}{n\pi} x \cos \frac{n\pi x}{L} + \frac{L^2}{n^2 \pi^2} \sin \frac{n\pi x}{L} \right]_0^{\frac{L}{2}} + \left[-\frac{L}{n\pi} (L-x) \cos \frac{n\pi x}{L} - \frac{L^2}{n^2 \pi^2} \sin \frac{n\pi x}{L} \right]_{\frac{L}{2}}^L \right\}$$



$$= \frac{4b}{L^2} \left\{ \left[\left(-\frac{L}{n\pi} \frac{L}{2} \cos \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right) - (0) \right]_{0}^{\frac{L}{2}} + \left[(0) - \left(-\frac{L}{n\pi} \left(\frac{L}{2} \right) \cos \frac{n\pi}{2} - \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right) \right]_{\frac{L}{2}}^L \right\}$$

$$= \frac{4b}{L^2} \left[\cancel{\frac{L^2}{2n\pi} \cos \frac{n\pi}{2}} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} + \cancel{\frac{L^2}{2n\pi} \cos \frac{n\pi}{2}} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$= \frac{4b}{L^2} \left[\frac{2L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$b_n = \frac{8b}{n^2\pi^2} \sin \frac{n\pi}{2}$$

Sub b_n in (4)

$$\therefore y(x,t) = \sum_{n=1}^{\infty} \frac{8b}{n^2\pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L}$$

$$\therefore y(x,t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l}$$

This is the required displacement.

3. A string of length l is fastened at both ends. The midpoint of the string is displaced transversely through a small distance ' b ' and the string is released from the rest in that position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.

Solution:

Replace L by l in the above problem

4. A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by

$y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement y at any distance x

from one end any time t .

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \text{ ----- (1)}$$



The Boundary and Initial conditions are

$$\text{i) } y(0, t) = 0$$

$$\text{ii) } y(l, t) = 0$$

$$\text{iii) } \frac{\partial y}{\partial t}(x, 0) = 0$$

$$\text{iv) } y(x, 0) = f(x) = y_0 \sin^3 \frac{\pi x}{l}, \quad 0 < x < l$$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0, t) = (A \cos 0 + B \sin 0)(C \cos pat + D \sin pat)$$

$$0 = (A)(C \cos pat + D \sin pat)$$

$$\text{Here } (C \cos pat + D \sin pat) \neq 0 \quad \therefore \boxed{A = 0}$$

Sub A=0 in (1)

$$(1) \Rightarrow y(x, t) = (B \sin px)(C \cos pat + D \sin pat) \text{ ----- (2)}$$

Applying condn (ii) in (2)

$$y(l, t) = (B \sin pl)(C \cos pat + D \sin pat)$$

$$0 = (B \sin pl)(C \cos pat + D \sin pat)$$

$$\text{Here } B \neq 0, (C \cos pat + D \sin pat) \neq 0$$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2) \Rightarrow y(x, t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + D \sin \frac{n\pi a}{l} t \right) \text{ ----- (3)}$$



Diff (3) partially w.r.to 't'

$$(2) \Rightarrow \frac{\partial y}{\partial t}(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left[-C \sin \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right) + D \cos \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right) \right]$$

Apply condn. (iii) in the above equation

$$\frac{\partial y}{\partial t}(x,0) = \left(B \sin \frac{n\pi}{l} x \right) \left[\cancel{-C \sin 0 \times \left(\frac{n\pi a}{l} \right)} + D \cos 0 \times \left(\frac{n\pi a}{l} \right) \right]$$

$$0 = \left(B \sin \frac{n\pi}{l} x \right) \left[D \times \left(\frac{n\pi a}{l} \right) \right]$$

Here $B \neq 0$, $\sin \frac{n\pi}{l} x \neq 0$, $\frac{n\pi a}{l} \neq 0$, $\therefore \boxed{D=0}$

Sub the value of D in (3)

$$(3) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + 0 \right)$$

$$y(x,t) = BC \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t$$

$$y(x,t) = b_1 \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \quad \text{let } BD = b_1$$

The most general solution is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t} \text{ ----- (4)}$$

Applying condn (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \quad \because \cos 0 = 1$$



$$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$$

$$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \quad \because \sin^3 \theta = \frac{1}{4} [3 \sin \theta - \sin 3\theta]$$

$$y_0 \left\{ \frac{1}{4} \left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right] \right\} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$$

$$\frac{3y_0}{4} \sin \frac{\pi x}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + b_4 \sin \frac{4\pi x}{l} + \dots$$

Equating co-efficients of likely terms on both sides

$$b_1 = \frac{3y_0}{4}; b_2 = 0; b_3 = \frac{y_0}{4}; b_4 = b_5 = b_6 \dots = 0.$$

Sub these values in (4)

$$(4) \Rightarrow y(x,t) = b_1 \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} + b_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi at}{l} + b_3 \sin \frac{3\pi x}{l} \cos \frac{3\pi at}{l} + \dots$$

$$y(x,t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi at}{l}$$

5. A tightly stretched string end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $\lambda(lx - x^2)$, then show that the displacement of given string is

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)\pi x}{l} \sin \frac{(2n-1)\pi at}{l}.$$

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \text{ ----- (1)}$$

The Boundary and Initial conditions are



$$\text{i) } y(0, t) = 0$$

$$\text{ii) } y(l, t) = 0$$

$$\text{iii) } y(x, 0) = 0$$

$$\text{iv) } \frac{\partial y}{\partial t}(x, 0) = \lambda(lx - x^2), 0 < x < l$$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0, t) = (A \cos 0 + \cancel{B \sin 0})(C \cos pat + D \sin pat)$$

$$0 = (A)(C \cos pat + D \sin pat)$$

$$\text{Here } (C \cos pat + D \sin pat) \neq 0 \therefore \boxed{A = 0}$$

Sub $A=0$ in (1)

$$(1) \Rightarrow y(x, t) = (B \sin px)(C \cos pat + D \sin pat) \text{ ----- (2)}$$

Applying condn (ii) in (2)

$$y(l, t) = (B \sin pl)(C \cos pat + D \sin pat)$$

$$0 = (B \sin pl)(C \cos pat + D \sin pat)$$

$$\text{Here } B \neq 0, (C \cos pat + D \sin pat) \neq 0$$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2) \Rightarrow y(x, t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + D \sin \frac{n\pi a}{l} t \right) \text{ ----- (3)}$$

Apply condn. (iii) in the above equation



$$(3) \Rightarrow y(x,0) = \left(B \sin \frac{n\pi}{l} x \right) (C \cos 0 + D \sin 0)$$

$$0 = \left(B \sin \frac{n\pi}{l} x \right) (C)$$

Here $B \neq 0$, $\sin \frac{n\pi}{l} x \neq 0$, $\therefore C = 0$

Sub the value of C in (3)

$$(3) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left(D \sin \frac{n\pi a}{l} t \right)$$

$$y(x,t) = BD \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t$$

$$y(x,t) = b_1 \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \quad \text{let } BD = b_1$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t \quad \text{----- (4)}$$

Diff (4) partially w.r.to 't'

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right)$$

Applying condn (iv) in the above eqn.

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos 0 \times \left(\frac{n\pi a}{l} \right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \times \left(\frac{n\pi a}{l} \right) \quad \because \cos 0 = 1$$

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \quad \text{let } B_n = b_n \frac{n\pi a}{l}$$



$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} x$$

Which is half range Fourier sine series in $(0, l)$

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l \lambda x(l-x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2\lambda}{l} \left[\cancel{(lx-x^2)} \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (l-2x) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) + (-2) \left(\frac{\cos \frac{n\pi x}{l}}{\frac{n^3 \pi^3}{l^3}} \right) \right]_0^l$$

$$= \frac{2\lambda}{l} \left[\frac{-2l^3}{n^3 \pi^3} \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-4\lambda l^3}{n^3 \pi^3} [\cos n\pi - \cos 0]$$

$$b_n \frac{n\pi a}{l} = \frac{-4\lambda l^3}{n^3 \pi^3} [(-1)^n - 1]$$

$$b_n = \frac{-4\lambda l^3}{n^4 \pi^4 a} [(-1)^n - 1]$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8\lambda l^3}{n^4 \pi^4 a} & \text{if } n \text{ is odd} \end{cases}$$

Sub b_n in (4)

$$y(x,t) = \sum_{n=1,3,5,\dots}^{\infty} \frac{8\lambda l^3}{n^4 \pi^4 a} \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t$$



(or)
$$y(x,t) = \frac{8\lambda l^3}{\pi^4 a} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)\pi}{l} x \sin \frac{(2n-1)\pi a}{l} t$$

6. If a string of length l is initially at rest in its equilibrium position whose ends are fixed and each of its points

is given a velocity v such that $v = \begin{cases} cx; & 0 < x < \frac{l}{2} \\ c(l-x); & \frac{l}{2} < x < l \end{cases}$, find the displacement of the string at any time t .

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \text{ ----- (1)}$$

The Boundary and Initial conditions are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $y(x,0) = 0$

iv) $\frac{\partial y}{\partial t}(x,0) = \begin{cases} cx; & 0 < x < \frac{l}{2} \\ c(l-x); & \frac{l}{2} < x < l \end{cases}$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0,t) = (A \cos 0 + B \sin 0)(C \cos pat + D \sin pat)$$

$$0 = (A)(C \cos pat + D \sin pat)$$

Here $(C \cos pat + D \sin pat) \neq 0 \Rightarrow A = 0$

Sub $A=0$ in (1)



$$(1) \Rightarrow y(x,t) = (B \sin px)(C \cos pat + D \sin pat) \text{ ----- (2)}$$

Applying condn (ii) in (2)

$$y(l,t) = (B \sin pl)(C \cos pat + D \sin pat)$$

$$0 = (B \sin pl)(C \cos pat + D \sin pat)$$

Here $B \neq 0$, $(C \cos pat + D \sin pat) \neq 0$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + D \sin \frac{n\pi a}{l} t \right) \text{ ----- (3)}$$

Apply condn. (iii) in the above equation

$$(3) \Rightarrow y(x,0) = \left(B \sin \frac{n\pi}{l} x \right) (C \cos 0 + \cancel{D \sin 0})$$

$$0 = \left(B \sin \frac{n\pi}{l} x \right) (C)$$

Here $B \neq 0$, $\sin \frac{n\pi}{l} x \neq 0$, $\therefore C = 0$

Sub the value of C in (3)

$$(3) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left(D \sin \frac{n\pi a}{l} t \right)$$

$$y(x,t) = BD \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t$$

$$y(x,t) = b_1 \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \quad \text{let } BD = b_1$$



The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t \quad \text{----- (4)}$$

Diff (4) partially w.r.to 't'

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right)$$

Applying condn (iv) in the above eqn.

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos 0 \times \left(\frac{n\pi a}{l} \right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \times \left(\frac{n\pi a}{l} \right) \quad \because \cos 0 = 1$$

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \quad \text{let } B_n = b_n \frac{n\pi a}{l}$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} x$$

Which is half range Fourier sine series in $(0,l)$

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \left[\int_0^{\frac{l}{2}} cx \sin \frac{n\pi x}{l} dx + \int_{\frac{l}{2}}^l c(l-x) \sin \frac{n\pi x}{l} dx \right] \quad \because f(x) = \begin{cases} cx; & 0 < x < \frac{l}{2} \\ c(l-x); & \frac{l}{2} < x < l \end{cases}$$

$$= \frac{2c}{l} \left[\int_0^{\frac{l}{2}} x \sin \frac{n\pi x}{l} dx + \int_{\frac{l}{2}}^l (l-x) \sin \frac{n\pi x}{l} dx \right]$$



$$\begin{aligned}
 &= \frac{2c}{l} \left\{ \left[(x) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) \right]_0^{\frac{l}{2}} + \left[(l-x) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (-1) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) \right]_{\frac{l}{2}}^l \right\} \\
 &= \frac{2c}{l} \left\{ \left[-\frac{l}{n\pi} x \cos \frac{n\pi x}{l} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi x}{l} \right]_0^{\frac{l}{2}} + \left[-\frac{l}{n\pi} (l-x) \cos \frac{n\pi x}{l} - \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi x}{l} \right]_{\frac{l}{2}}^l \right\} \\
 &= \frac{2c}{l} \left\{ \left[\left(-\frac{l}{n\pi} \frac{l}{2} \cos \frac{n\pi}{2} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} \right) - (0) \right]_0^{\frac{l}{2}} + \left[(0) - \left(-\frac{l}{n\pi} \left(\frac{l}{2} \right) \cos \frac{n\pi}{2} - \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} \right) \right]_{\frac{l}{2}}^l \right\} \\
 &= \frac{2c}{l} \left[\cancel{\frac{l^2}{2n\pi} \cos \frac{n\pi}{2}} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} + \cancel{\frac{l^2}{2n\pi} \cos \frac{n\pi}{2}} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} \right] \\
 &= \frac{2c}{l} \left[\frac{2l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} \right]
 \end{aligned}$$

$$b_n \frac{n\pi a}{l} = \frac{4cl}{n^2 \pi^2} \sin \frac{n\pi}{2} \quad \therefore B_n = b_n \frac{n\pi a}{l}$$

$$\boxed{b_n = \frac{4cl^2}{n^3 \pi^3 a} \sin \frac{n\pi}{2}}$$

Sub b_n in (4)

$$\therefore y(x,t) = \sum_{n=1}^{\infty} \frac{4cl^2}{n^3 \pi^3 a} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

$$\therefore y(x,t) = \frac{4cl^2}{\pi^3 a} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

7. If a string of length of l is initially at rest in its equilibrium position and each of its point is given the velocity $\frac{\partial y}{\partial t}(x,0) = V_0 \sin^3 \frac{\pi x}{l}$; $0 < x < l$. Determine the displacement function $y(x,t)$.

Solution:



One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \text{ ----- (1)}$$

The Boundary and Initial conditions are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $y(x,0) = 0$

iv) $\frac{\partial y}{\partial t}(x,0) = V_0 \sin^3 \frac{\pi x}{l}, 0 < x < l$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0,t) = (A \cos 0 + B \sin 0)(C \cos pat + D \sin pat)$$

$$0 = (A)(C \cos pat + D \sin pat)$$

Here $(C \cos pat + D \sin pat) \neq 0 \therefore \boxed{A=0}$

Sub $A=0$ in (1)

$$(1) \Rightarrow y(x,t) = (B \sin px)(C \cos pat + D \sin pat) \text{ ----- (2)}$$

Applying condn (ii) in (2)

$$y(l,t) = (B \sin pl)(C \cos pat + D \sin pat)$$

$$0 = (B \sin pl)(C \cos pat + D \sin pat)$$

Here $B \neq 0, (C \cos pat + D \sin pat) \neq 0$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$



Sub the value of p in (2)

$$(2) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + D \sin \frac{n\pi a}{l} t \right) \text{ ----- (3)}$$

Apply condn. (iii) in the above equation

$$(3) \Rightarrow y(x,0) = \left(B \sin \frac{n\pi}{l} x \right) (C \cos 0 + \cancel{D \sin 0})$$

$$0 = \left(B \sin \frac{n\pi}{l} x \right) (C)$$

Here $B \neq 0$, $\sin \frac{n\pi}{l} x \neq 0$, $\therefore \boxed{C=0}$

Sub the value of C in (3)

$$(3) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left(D \sin \frac{n\pi a}{l} t \right)$$

$$y(x,t) = BD \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t$$

$$y(x,t) = b_1 \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \quad \text{let } BD = b_1$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t \text{ ----- (4)}$$

Diff (4) partially w.r.to 't'

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right)$$

Applying condn (iv) in the above eqn.

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos 0 \times \left(\frac{n\pi a}{l} \right)$$



$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \times \left(\frac{n\pi a}{l} \right) \quad \because \cos 0 = 1$$

$$V_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l}$$

$$V_0 \left\{ \frac{1}{4} \left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right] \right\} = \sum_{n=1}^{\infty} b_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l} \quad \because \sin^3 \theta = \frac{1}{4} [3 \sin \theta - \sin 3\theta]$$

$$\frac{3V_0}{4} \sin \frac{\pi x}{l} - \frac{V_0}{4} \sin \frac{3\pi x}{l} = b_1 \frac{\pi a}{l} \sin \frac{\pi x}{l} + b_2 \frac{2\pi a}{l} \sin \frac{2\pi x}{l} + b_3 \frac{3\pi a}{l} \sin \frac{3\pi x}{l} + b_4 \frac{4\pi a}{l} \sin \frac{4\pi x}{l} + \dots$$

Equating co-efficients of likely terms on both sides

$$b_1 \frac{\pi a}{l} = \frac{3V_0}{4}; b_2 \frac{2\pi a}{l} = 0; b_3 \frac{3\pi a}{l} = \frac{-V_0}{4}; b_4 = b_5 = b_6 \dots = 0.$$

$$b_1 = \frac{3V_0 l}{4\pi a}; b_2 = 0; b_3 = \frac{-V_0 l}{12\pi a}; b_4 = b_5 = b_6 \dots = 0.$$

Sub these values in (4)

$$(4) \Rightarrow y(x,t) = b_1 \sin \frac{\pi x}{l} \cos \frac{\pi a t}{l} + b_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi a t}{l} + b_3 \sin \frac{3\pi x}{l} \cos \frac{3\pi a t}{l} + \dots$$

$$y(x,t) = \frac{3V_0 l}{4\pi a} \sin \frac{\pi x}{l} \cos \frac{\pi a t}{l} - \frac{V_0 l}{12\pi a} \sin \frac{3\pi x}{l} \cos \frac{3\pi a t}{l}$$



UNIT III APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

3.2 ONE DIMENSIONAL HEAT EQUATION

One dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } \alpha^2 = \frac{k}{\rho c} = \frac{\text{Thermal conductivity}}{\text{density} \times \text{Specific heat capacity}}$$

$u(x, t) \rightarrow$ The temperature distribution at any point x from one end at time t .

The various Possible Solution of 1-D heat equation.

(i) $u(x, t) = (Ae^{px} + Be^{-px})Ce^{\alpha^2 p^2 t}$

(ii) $u(x, t) = (A \cos px + B \sin px)Ce^{-\alpha^2 p^2 t}$

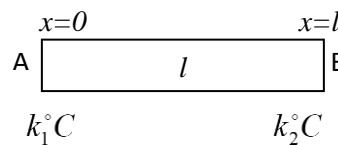
(iii) $u(x, t) = Ax + B$

The boundary and initial conditions.

i) $u(0, t) = k_1^\circ C$

ii) $u(l, t) = k_2^\circ C$

iii) $u(x, 0) = f(x)$



The correct solution is $u(x, t) = (A \cos px + B \sin px)Ce^{-\alpha^2 p^2 t}$

The steady state solution in 1-D heat equation:

Solution:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{-----(1)}$$

In steady state $t=0$ then $\frac{\partial u}{\partial t} = 0$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$$

Integrating twice we get $u(x) = Ax + B$



A rod of length l has its ends A and B are kept at 0°C and 100°C until steady state condition prevail. If the temperature at B is reduced suddenly to 0°C and kept so while that of A is maintained. Find the temperature $u(x, t)$ at a distance x from A and at time t .

Solution:

The 1-D heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ where $\alpha^2 = \frac{k}{\rho c}$

To find steady state solution $u(x, 0) = u(x)$

In steady state $t=0$ then $\frac{\partial u}{\partial t} = 0$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$$

Integrating twice we get $\boxed{u(x) = Ax + B}$ -----(1)

The boundary conditions are i) $u(0) = 0^\circ\text{C}$ ii) $u(l) = 100^\circ\text{C}$

Applying condn (i) in (1)

$$(1) \Rightarrow u(0) = 0 + B \Rightarrow \boxed{B = 0}$$

Sub B in (1)

$$u(x) = Ax \text{ -----(2)}$$

Applying condn (ii) in (2)

$$u(l) = Al \Rightarrow 100 = Al \Rightarrow \boxed{A = \frac{100}{l}}$$

Sub A in (2)

$$\boxed{u(x) = \frac{100x}{l}}$$

The boundary and initial conditions are

$$\text{i) } u(0, t) = 0^\circ\text{C}$$



$$\text{ii) } u(l, t) = 100^\circ \text{C}$$

$$\text{iii) } u(x, 0) = f(x) = \frac{100x}{l}, \quad 0 \leq x \leq l$$

The correct solution is

$$u(x, t) = (A \cos px + B \sin px) C e^{-\alpha^2 p^2 t} \text{ ----- (1)}$$

Apply condn. (i) in (1)

$$u(0, t) = (A \cos 0 + \cancel{B \sin 0}) C e^{-\alpha^2 p^2 t}$$

$$0 = A C e^{-\alpha^2 p^2 t}$$

$$\text{Here } C \neq 0, e^{-\alpha^2 p^2 t} \neq 0 \therefore \boxed{A = 0}$$

Sub A in (1)

$$u(x, t) = (B \sin px) C e^{-\alpha^2 p^2 t} \text{ ----- (2)}$$

Apply condn. (ii) in (2)

$$u(l, t) = (B \sin pl) C e^{-\alpha^2 p^2 t}$$

$$0 = (B \sin pl) C e^{-\alpha^2 p^2 t}$$

$$\text{Here } B \neq 0, C \neq 0, e^{-\alpha^2 p^2 t} \neq 0 \therefore \boxed{\sin pl = 0}$$

$$\sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub p in (2)

$$u(x, t) = \left(B \sin \frac{n\pi x}{l} \right) C e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$u(x, t) = b_1 \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \quad BC = b_1 \text{ (say)}$$

The most general solution is



$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \quad \text{----- (3)}$$

Apply condn (iii) in (3)

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-0}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \because e^{-0} = 1$$

This is Fourier sine series of $f(x)$ in $(0,l)$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2} \int_0^l x \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2} \left[(x) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) \right]_0^l$$

$$= \frac{200}{l^2} \left(\frac{-l}{n\pi} \right) \left[x \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-200}{ln\pi} [l \cos n\pi - 0]$$

$$= \frac{-200(-1)^n}{n\pi}$$

$$b_n = \frac{200}{n\pi} (-1)^{n+1}$$

Sub b_n in (3)



$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$u(x,t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

This is the required temperature.

The ends A and B of a rod l cm long have their temperatures kept at $30^\circ C$ and $80^\circ C$, until steady state conditions prevail. The temperature of the end B is suddenly reduced to $60^\circ C$ and that of A is increased to $40^\circ C$. Find the steady state temperature distribution in the rod after time t .

Solution:

The 1-D heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ where $\alpha^2 = \frac{k}{\rho c}$

To find steady state solution 1 $u(x,0) = u(x)$

In steady state $t=0$ then $\frac{\partial u}{\partial t} = 0$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$$

Integrating twice we get $u(x) = Ax + B$ -----(1)

The boundary conditions are i) $u(0) = 30^\circ C$ ii) $u(l) = 80^\circ C$

Applying condn (i) in (1)

$$(1) \Rightarrow u(0) = 0 + B \Rightarrow B = 30$$

Sub B in (1)

$$u(x) = Ax + 30$$
 -----(2)

Applying condn (ii) in (2)

$$u(l) = Al + 30 \Rightarrow 80 = Al + 30 \Rightarrow A = \frac{50}{l}$$

Sub A in (2)



$$u(x) = \frac{50x}{l} + 30 = f(x)$$

This $u(x)$ will be treated as the initial conditions $u(x, 0) = f(x)$

To find steady state solution $2 u(x, 0) = u(x)$

Integrating twice we get $u_t(x) = Ax + B$ -----(3)

The boundary conditions are i) $u_t(0) = 40^\circ C$ ii) $u_t(l) = 60^\circ C$

Applying condn (i) in (3)

$$(3) \Rightarrow u_t(0) = 0 + B \Rightarrow B = 40$$

Sub B in (1)

$$u_t(x) = Ax + 40$$
 -----(4)

Applying condn (ii) in (4)

$$u_t(l) = Al + 40 \Rightarrow 60 = Al + 40 \Rightarrow A = \frac{20}{l}$$

Sub A in (2)

$$u_t(x) = \frac{20x}{l} + 40$$

This $u_t(x)$ will be treated as the transient state temperature.

The required temperature is

$$u(x, t) = u_t(x, 0) + (A \cos px + B \sin px) C e^{-\alpha^2 p^2 t}$$

$$u(x, t) = \frac{20x}{l} + 40 + (A \cos px + B \sin px) C e^{-\alpha^2 p^2 t}$$
 ----- (5)

The boundary and initial conditions are

i) $u(0, t) = 40^\circ C$

ii) $u(l, t) = 60^\circ C$



$$\text{iii) } u(x, 0) = f(x) = \frac{50x}{l} + 30, \quad 0 \leq x \leq l$$

Apply condn (i) in (5)

$$u(0, t) = 0 + 40 + (A \cos 0 + B \sin 0) C e^{-\alpha^2 p^2 t}$$

$$40 = 0 + 40 + (A \cos 0 + \cancel{B \sin 0}) C e^{-\alpha^2 p^2 t}$$

$$0 = A C e^{-\alpha^2 p^2 t}$$

This $C \neq 0, e^{-\alpha^2 p^2 t} \neq 0 \therefore \boxed{A = 0}$

Sub A in (5)

$$u(x, t) = \frac{20x}{l} + 40 + (B \sin px) C e^{-\alpha^2 p^2 t} \text{ ----- (6)}$$

Apply condn (ii) in (6)

$$u(l, t) = 20 + 40 + (B \sin pl) C e^{-\alpha^2 p^2 t}$$

$$60 = 20 + 40 + (B \sin pl) C e^{-\alpha^2 p^2 t}$$

$$0 = (B \sin pl) C e^{-\alpha^2 p^2 t}$$

$$B \neq 0, C \neq 0, e^{-\alpha^2 p^2 t} \neq 0 \therefore \sin pl = 0$$

$$\sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub p in (6)

$$u(x, t) = \frac{20x}{l} + 40 + \left(B \sin \frac{n\pi x}{l} \right) C e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}}$$

$$u(x, t) = \frac{20x}{l} + 40 + b_1 \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}} \quad \therefore BC = b_1$$

The most general solution is



$$u(x,t) = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}} \text{ ----- (7)}$$

Apply condn (iii) in (7)

$$u(x,0) = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-0}$$

$$\frac{50x}{l} + 30 = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\frac{30x}{l} - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

To find b_n :

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l \left(\frac{30x}{l} - 10 \right) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \left[\left(\frac{30x}{l} - 10 \right) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - \left(\frac{30}{l} \right) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) \right]_0^l$$

$$= \frac{2}{l} \left[\left(\frac{-l}{n\pi} \right) \left(\frac{30x}{l} - 10 \right) \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-2}{n\pi} \left[(20) \cos n\pi + 10 \right]$$

$$\boxed{b_n = \frac{-20}{n\pi} \left[2(-1)^n + 1 \right]}$$

Sub b_n in (7)

$$u(x,t) = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} \frac{-20}{n\pi} \left[2(-1)^n + 1 \right] \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}}$$



$$u(x,t) = \frac{20x}{l} + 40 - \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [2(-1)^n + 1] \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}}$$



UNIT III APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

3.3 TWO DIMENSIONAL HEAT EQUATION (LAPLACE EQUATION)

- 1) The 2-D heat equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
- 2) The various possible solution of 2-D heat equation is
- i) $u(x, y) = (A \cos px + B \sin px) (C e^{py} + e^{-py})$
 - ii) $u(x, y) = (Ae^{px} + Be^{-px}) (C \cos py + \sin py)$
 - iii) $u(x, y) = (Ax+B) (Cy+D)$

1. A square plate is bounded by the lines $x=0, x=l, y=0$ and $y=l$, its faces are insulated. The temperature along upper horizontal edge is given by $u = x(l-x)$ when $0 < x < l$. while the other three edges are kept at $0^\circ C$. Find steady state solution in the plate.

Solution:

The 2-D heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The Boundary conditions are

- i) $u(0, y) = 0$
- ii) $u(l, y) = 0$
- iii) $u(x, 0) = 0$
- iv) $u(x, l) = f(x) = x(l-x), 0 \leq x \leq l$

Here the non zero temperature is parallel to x axis then the Correct solution is

$$u(x, y) = (A \cos px + B \sin px) (C e^{py} + D e^{-py}) \text{ ----- (1)}$$

Apply condn (i) in (1)

$$u(0, y) = (A \cos 0 + B \sin 0) (C e^{py} + D e^{-py})$$

$$0 = A (C e^{py} + D e^{-py})$$

Here $C e^{py} + D e^{-py} \neq 0 \therefore A = 0$

Sub A in (1)

$$u(x, y) = (B \sin px) (C e^{py} + D e^{-py}) \text{ ----- (2)}$$

Apply condn (ii) in (2)

$$u(l, y) = (B \sin pl) (C e^{py} + D e^{-py})$$

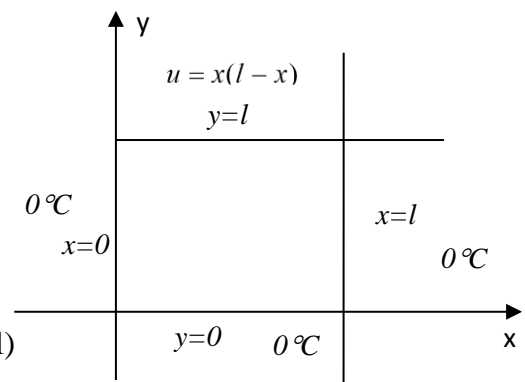
$$0 = (B \sin pl) (C e^{py} + D e^{-py})$$

Here $C e^{py} + D e^{-py} \neq 0, B \neq 0 \therefore \sin pl = 0$

$$\sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow p = \frac{n\pi}{l}$$

Sub p in (2)

$$u(x, y) = \left(B \sin \frac{n\pi x}{l} \right) \left(C e^{\frac{n\pi y}{l}} + D e^{-\frac{n\pi y}{l}} \right) \text{ ----- (3)}$$





Apply condn (iii) in (3)

$$u(x, 0) = \left(B \sin \frac{n\pi x}{l} \right) (Ce^0 + De^{-0})$$

$$0 = \left(B \sin \frac{n\pi x}{l} \right) (C + D)$$

Here $\sin \frac{n\pi x}{l} \neq 0, B \neq 0, \therefore C + D = 0 \Rightarrow D = -C$

Sub $D = -C$ in (3)

$$u(x, y) = \left(B \sin \frac{n\pi x}{l} \right) \left(Ce^{\frac{n\pi y}{l}} - Ce^{-\frac{n\pi y}{l}} \right)$$

$$u(x, y) = BC \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right)$$

$$u(x, y) = BC \sin \frac{n\pi x}{l} \left(2 \sinh \frac{n\pi y}{l} \right) \because \frac{e^\theta - e^{-\theta}}{2} = \sinh \theta$$

$$u(x, y) = b_1 \sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l} \quad \text{let } 2BC = b_1$$

The most general Solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l} \quad \text{----- (4)}$$

Apply condn (iv) in (4)

$$u(x, l) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sinh n\pi$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \quad \text{let } B_n = b_n \sinh n\pi$$

This is Fourier sine series in $(0, l)$

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l x(l-x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \left[\cancel{(lx-x^2)} \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (l-2x) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) + (-2) \left(\frac{\cos \frac{n\pi x}{l}}{\frac{n^3 \pi^3}{l^3}} \right) \right]_0^l$$

$$= \frac{2}{l} \left[\frac{-2l^3}{n^3 \pi^3} \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-4l^3}{ln^3 \pi^3} [\cos n\pi - \cos 0]$$

$$b_n \sinh n\pi = \frac{-4l^2}{n^3 \pi^3} [(-1)^n - 1]$$



$$b_n = \frac{-4l^2}{n^3 \pi^3 \sinh n\pi} [(-1)^n - 1]$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8l^2}{n^3 \pi^3 \sinh n\pi} & \text{if } n \text{ is odd} \end{cases}$$

Sub b_n in (4)

$$u(x, y) = \sum_{n=1,3,5,\dots}^{\infty} \frac{8l^2}{n^3 \pi^3 \sinh n\pi} \sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l}$$

$$u(x, y) = \frac{8l^2}{\pi^3} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3 \sinh n\pi} \sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l}$$

2. A rectangular plate with insulated surface is 20cm wide and so long compared to its width that it may be considered infinite in length without introducing appreciable error. The temperature at short edge $x = 0$ is given by $u = \begin{cases} 10y, & 0 \leq y \leq 10 \\ 10(20 - y), & 10 \leq y \leq 20 \end{cases}$ and the two long edges as well as the other short edges are kept at 0°C . Find the steady state temperature distribution in the plate.

Solution:

The 2-D heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The Boundary conditions are

i) $u(x, 0) = 0$

ii) $u(x, 20) = 0$

iii) $u(\infty, y) = 0$

iv) $u(0, y) = \begin{cases} 10y, & 0 \leq y \leq 10 \\ 10(20 - y), & 10 < y \leq 20 \end{cases}$

The correct solution is

$$u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py) \text{ ----- (1)}$$

Apply condn (i) in (1)

$$u(x, 0) = (Ae^{px} + Be^{-px})(C \cos 0 + D \sin 0)$$

$$0 = (Ae^{px} + Be^{-px})C$$

Here $(Ae^{px} + Be^{-px}) \neq 0, \therefore C = 0$

Sub C in (1)

$$u(x, y) = (Ae^{px} + Be^{-px})(D \sin py) \text{ ----- (2)}$$

Apply condn (ii) in (2)

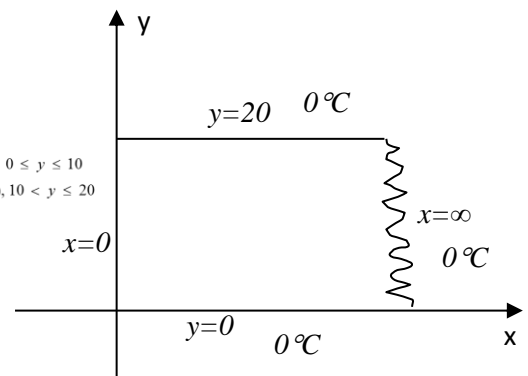
$$u(x, 20) = (Ae^{px} + Be^{-px})(D \sin 20p)$$

$$0 = (Ae^{px} + Be^{-px})(D \sin 20p)$$

Here $(Ae^{px} + Be^{-px}) \neq 0, D \neq 0$

$$\therefore \sin 20p = 0 \Rightarrow \sin 20p = \sin n\pi \Rightarrow 20p = n\pi \Rightarrow p = \frac{n\pi}{20}$$

$$u(0, y) = \begin{cases} 10y, & 0 \leq y \leq 10 \\ 10(20 - y), & 10 < y \leq 20 \end{cases}$$





Sub p in (2)

$$(2) \Rightarrow u(x, y) = \left(Ae^{\frac{n\pi x}{20}} + Be^{-\frac{n\pi x}{20}} \right) \left(D \sin \frac{n\pi y}{20} \right) \text{----- (3)}$$

Apply condn (iii) in (3)

$$u(\infty, y) = \left(Ae^{\infty} + Be^{-\infty} \right) \left(D \sin \frac{n\pi y}{20} \right)$$

$$0 = \left(Ae^{\infty} \right) \left(D \sin \frac{n\pi y}{20} \right)$$

temperature $e^{\infty} \neq 0, D \neq 0, \sin \frac{n\pi y}{20} \neq 0, \therefore \boxed{A = 0}$

sub A in (3)

$$(3) \Rightarrow u(x, y) = \left(Be^{-\frac{n\pi x}{20}} \right) \left(D \sin \frac{n\pi y}{20} \right)$$

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{20} e^{-\frac{n\pi x}{10}} \text{----- (4)}$$

Apply condn (iv) in (4)

$$u(0, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{20} e^{-0}$$

$$f(y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{20}$$

This is half range sine series in $(0, 20)$

$$b_n = \frac{2}{l} \int_0^l f(y) \sin \frac{n\pi y}{l} dy$$

$$= \frac{2}{20} \int_0^{20} f(y) \sin \frac{n\pi y}{20} dy$$

$$= \frac{1}{10} \left[\int_0^{10} 10y \sin \frac{n\pi y}{20} dy + \int_{10}^{20} 10(20 - y) \sin \frac{n\pi y}{20} dy \right]$$

$$= \frac{10}{10} \left[\int_0^{10} y \sin \frac{n\pi y}{20} dy + \int_{10}^{20} (20 - y) \sin \frac{n\pi y}{20} dy \right]$$

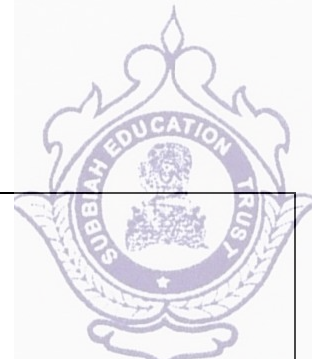
$$= \left[(y) \left(\frac{-\cos \frac{n\pi y}{20}}{\frac{n\pi}{20}} \right) - (1) \left(\frac{-\sin \frac{n\pi y}{20}}{\frac{n^2 \pi^2}{400}} \right) \right]_0^{10} + \left[(20 - y) \left(\frac{-\cos \frac{n\pi y}{20}}{\frac{n\pi}{20}} \right) - (-1) \left(\frac{-\sin \frac{n\pi y}{20}}{\frac{n^2 \pi^2}{400}} \right) \right]_{10}^{20}$$

$$= \left[-\frac{20}{n\pi} (y) \cos \frac{n\pi y}{20} + \frac{400}{n^2 \pi^2} \sin \frac{n\pi y}{20} \right]_0^{10} + \left[-\frac{20}{n\pi} (20 - y) \cos \frac{n\pi y}{20} - \frac{400}{n^2 \pi^2} \sin \frac{n\pi y}{20} \right]_{10}^{20}$$

$$= \left[\left(-\frac{20}{n\pi} (10) \cos \frac{n\pi}{2} + \frac{400}{n^2 \pi^2} \sin \frac{n\pi}{2} \right) - (0) \right] + \left[(0) - \left(-\frac{20}{n\pi} (10) \cos \frac{n\pi}{2} - \frac{400}{n^2 \pi^2} \sin \frac{n\pi}{2} \right) \right]$$

$$= -\frac{200}{n\pi} \cos \frac{n\pi}{2} + \frac{400}{n^2 \pi^2} \sin \frac{n\pi}{2} + \frac{200}{n\pi} \cos \frac{n\pi}{2} + \frac{400}{n^2 \pi^2} \sin \frac{n\pi}{2}$$

$$\boxed{b_n = \frac{800}{n^2 \pi^2} \sin \frac{n\pi}{2}}$$



Sub b_n in (4)

$$u(x, y) = \sum_{n=1}^{\infty} \frac{800}{n^2 \pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi y}{20} e^{-\frac{n\pi x}{10}} \quad \boxed{u(x, y) = \frac{800}{n^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi y}{20} e^{-\frac{n\pi x}{10}}}$$

3. An infinitely long rectangular plate is of width 10cm. The temperature along the short edge $y=0$ is given by

$$u = \begin{cases} 20x, & 0 \leq x \leq 5 \\ 20(10-x), & 5 \leq x \leq 10 \end{cases} \cdot \text{If all the other edges are kept at zero temperature. Find the steady state temperature at any point on it.}$$

Solution:

The 2-D heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

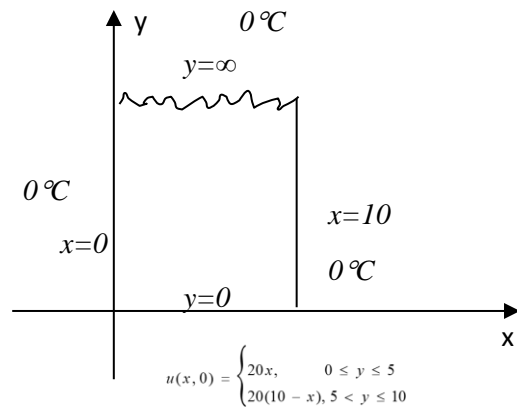
The Boundary conditions are

i) $u(0, y) = 0$

ii) $u(10, y) = 0$

iii) $u(x, \infty) = 0$

iv) $u(x, 0) = \begin{cases} 20x, & 0 \leq x \leq 5 \\ 20(10-x), & 5 < x \leq 10 \end{cases}$



Here the non zero boundary condition is parallel to x axis then

The correct solution is

$$u(x, y) = (A \cos px + B \sin px)(C e^{py} + D e^{-py}) \text{ ----- (1)}$$

Apply condn (i) in (1)

$$u(0, y) = (A \cos 0 + B \sin 0)(C e^{py} + D e^{-py})$$

$$0 = A(C e^{py} + D e^{-py})$$

Here $(C e^{py} + D e^{-py}) \neq 0, \therefore \boxed{A=0}$

Sub C in (1)

$$u(x, y) = (B \sin px)(C e^{py} + D e^{-py}) \text{ ----- (2)}$$

Apply condn (ii) in (2)

$$u(10, y) = (B \sin 10p)(C e^{py} + D e^{-py})$$

$$0 = (B \sin 10p)(C e^{py} + D e^{-py})$$

Here $B \neq 0, (C e^{py} + D e^{-py}) \neq 0 \therefore \sin 10p = 0$

$$\therefore \sin 10p = 0 \Rightarrow \sin 10p = \sin n\pi \Rightarrow 10p = n\pi \Rightarrow \boxed{p = \frac{n\pi}{10}}$$

Sub p in (2)

$$u(x, y) = \left(B \sin \frac{n\pi x}{10} \right) \left(C e^{\frac{n\pi y}{10}} + D e^{-\frac{n\pi y}{10}} \right) \text{ ----- (3)}$$

Apply condn (iii) in (3)

$$u(x, \infty) = \left(B \sin \frac{n\pi x}{10} \right) (C e^{\infty} + D e^{-\infty})$$

$$0 = \left(B \sin \frac{n\pi x}{10} \right) (C e^{\infty} + D e^{-\infty})$$

Here $B \neq 0, e^{\infty} \neq 0, \sin \frac{n\pi x}{10} \neq 0 \therefore C = 0$



sub C in (3)

$$u(x, y) = \left(B \sin \frac{n\pi x}{10} \right) \left(D e^{-\frac{n\pi y}{10}} \right)$$

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} e^{-\frac{n\pi y}{10}} \text{ ----- (4)}$$

Apply condn (iv) in (4)

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} e^{-0}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10}$$

This is half range sine series in (0,10)

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{10} \int_0^{10} f(x) \sin \frac{n\pi x}{10} dx$$

$$= \frac{1}{10} \left[\int_0^5 20x \sin \frac{n\pi x}{20} dx + \int_5^{10} 20(10-x) \sin \frac{n\pi x}{20} dx \right]$$

$$= \frac{20}{10} \left[\int_0^5 x \sin \frac{n\pi x}{10} dx + \int_5^{10} (10-x) \sin \frac{n\pi x}{10} dx \right]$$

$$= 2 \left\{ \left[x \left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{10}}{\frac{n^2 \pi^2}{100}} \right) \right]_0^5 + \left[(10-x) \left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} \right) - (-1) \left(\frac{-\sin \frac{n\pi x}{10}}{\frac{n^2 \pi^2}{100}} \right) \right]_5^{10} \right\}$$

$$= 2 \left\{ \left[-\frac{10}{n\pi} (x) \cos \frac{n\pi x}{10} + \frac{100}{n^2 \pi^2} \sin \frac{n\pi x}{10} \right]_0^5 + \left[-\frac{10}{n\pi} (10-x) \cos \frac{n\pi x}{10} - \frac{100}{n^2 \pi^2} \sin \frac{n\pi x}{10} \right]_5^{10} \right\}$$

$$= 2 \left\{ \left[\left(-\frac{10}{n\pi} (5) \cos \frac{n\pi}{2} + \frac{100}{n^2 \pi^2} \sin \frac{n\pi}{2} \right) - (0) \right] + \left[(0) - \left(-\frac{10}{n\pi} (5) \cos \frac{n\pi}{2} - \frac{100}{n^2 \pi^2} \sin \frac{n\pi}{2} \right) \right] \right\}$$

$$= 2 \left[-\frac{50}{n\pi} \cos \frac{n\pi}{2} + \frac{100}{n^2 \pi^2} \sin \frac{n\pi}{2} + \frac{50}{n\pi} \cos \frac{n\pi}{2} + \frac{100}{n^2 \pi^2} \sin \frac{n\pi}{2} \right]$$

$$b_n = \frac{400}{n^2 \pi^2} \sin \frac{n\pi}{2}$$

Sub b_n in (4)

$$u(x, y) = \sum_{n=1}^{\infty} \frac{400}{n^2 \pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{10} e^{-\frac{n\pi y}{10}} \Rightarrow u(x, y) = \frac{400}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{10} e^{-\frac{n\pi y}{10}}$$

- 4. An infinite long rectangular plate with insulated surfaces is 10 cm wide. The two long edges and one short edge are kept at 0°C, while the other short edge $x = 0$ is kept at temperature $u = \begin{cases} 20y, & 0 \leq y \leq 5 \\ 20(10-y), & 5 < y \leq 10 \end{cases}$.**

Find the steady state temperature distribution in the plate.

Solution:

The 2-D heat equation is



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The Boundary conditions are

i) $u(x, 0) = 0$

ii) $u(x, 10) = 0$

iii) $u(\infty, y) = 0$

iv) $u(0, y) = \begin{cases} 20y, & 0 \leq y \leq 5 \\ 20(10 - y), & 5 < y \leq 10 \end{cases}$

The correct solution is

$$u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py) \text{ ----- (1)}$$

Apply condn (i) in (1)

$$u(x, 0) = (Ae^{px} + Be^{-px})(C \cos 0 + D \sin 0)$$

$$0 = (Ae^{px} + Be^{-px})C$$

Here $(Ae^{px} + Be^{-px}) \neq 0, \therefore C = 0$

Sub C in (1)

$$u(x, y) = (Ae^{px} + Be^{-px})(D \sin py) \text{ ----- (2)}$$

Apply condn (ii) in (2)

$$u(x, 10) = (Ae^{px} + Be^{-px})(D \sin 10p)$$

$$0 = (Ae^{px} + Be^{-px})(D \sin 10p)$$

Here $(Ae^{px} + Be^{-px}) \neq 0, D \neq 0$

$$\therefore \sin 10p = 0 \Rightarrow \sin 10p = 0 \sin n\pi \Rightarrow 10p = n\pi \Rightarrow p = \frac{n\pi}{10}$$

Sub p in (2)

$$(2) \Rightarrow u(x, y) = \left(Ae^{\frac{n\pi x}{10}} + Be^{-\frac{n\pi x}{10}} \right) \left(D \sin \frac{n\pi y}{10} \right) \text{ ----- (3)}$$

Apply condn (iii) in (3)

$$u(\infty, y) = (Ae^\infty + Be^{-\infty}) \left(D \sin \frac{n\pi y}{10} \right)$$

$$0 = (Ae^\infty) \left(D \sin \frac{n\pi y}{10} \right)$$

temperature $e^\infty \neq 0, D \neq 0, \sin \frac{n\pi y}{10} \neq 0, \therefore A = 0$

sub A in (3)

$$(3) \Rightarrow u(x, y) = \left(Be^{-\frac{n\pi x}{10}} \right) \left(D \sin \frac{n\pi y}{10} \right)$$

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{10} e^{-\frac{n\pi x}{10}} \text{ ----- (4)}$$

Apply condn (iv) in (4)

$$u(0, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{10} e^{-0}$$

$$f(y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{10}$$

Which is half range sine series in (0,10)



$$b_n = \frac{2}{l} \int_0^l f(y) \sin \frac{n\pi y}{10} dy$$

$$b_n = \frac{2}{10} \int_0^{10} f(y) \sin \frac{n\pi y}{10} dy$$

$$b_n = \frac{2}{10} \left[\int_0^5 20y \sin \frac{n\pi y}{10} dy + \int_5^{10} 20(10-y) \sin \frac{n\pi y}{10} dy \right]$$

$$b_n = \frac{40}{10} \left[\int_0^5 y \sin \frac{n\pi y}{10} dy + \int_5^{10} (10-y) \sin \frac{n\pi y}{10} dy \right]$$

$$= \frac{20}{5} \left[\left(y \left(\frac{-\cos \frac{n\pi y}{10}}{\frac{n\pi}{10}} \right) - (1) \left(\frac{-\sin \frac{n\pi y}{10}}{\frac{n^2 \pi^2}{100}} \right) \right) \Big|_0^5 + \left((10-x) \left(\frac{-\cos \frac{n\pi y}{10}}{\frac{n\pi}{10}} \right) - (-1) \left(\frac{-\sin \frac{n\pi y}{10}}{\frac{n^2 \pi^2}{100}} \right) \right) \Big|_5^{10} \right]$$

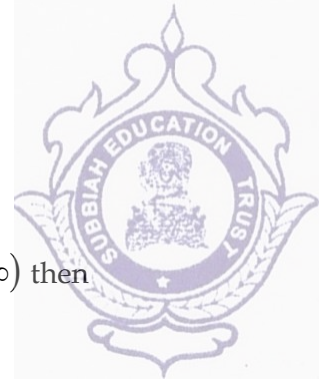
$$b_n = 4 \left[\frac{-50}{n\pi} \cos \frac{n\pi}{2} + \frac{100}{n^2 \pi^2} \sin \frac{n\pi}{2} + \frac{50}{n\pi} \cos \frac{n\pi}{2} + \frac{100}{n^2 \pi^2} \sin \frac{n\pi}{2} \right]$$

$$b_n = \frac{800}{n^2 \pi^2} \sin \frac{n\pi}{2}$$

Sub b_n in (4)

$$u(x, y) = \sum_{n=1}^{\infty} \frac{800}{n^2 \pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi y}{10} e^{-\frac{n\pi x}{10}}$$

$$u(x, y) = \frac{800}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi y}{10} e^{-\frac{n\pi x}{10}}$$



4.1 FOURIER TRANSFORMS PAIR

1. State Fourier integral theorem.

Solution :

If $f(x)$ is piecewise continuous, differentiable and absolutely integrable in $(-\infty, \infty)$ then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{is(x-t)} dt ds$$

2. If $F(s)$ is the Fourier transform of $f(x)$, then show that $F\{f(x-a)\} = e^{ias} F(s)$ **Solution :**

$$\text{Given } F[f(x)] = F(s)$$

The Fourier Transform of $f(x)$ is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$$

$$\text{Let } x-a = t \Rightarrow dx = dt$$

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is(t+a)} dt \\ &= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt \end{aligned}$$

$$\boxed{F[f(x-a)] = e^{ias} F[f(x)]}$$

3. State Convolution theorem in Fourier Transform.

Solution :

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms .

$$\text{i.e. } F[f(x) * g(x)] = F[f(x)] F[g(x)] = F(s).G(s)$$

4. If $F\{f(x)\} = F(s)$, then find $F\{e^{iax} f(x)\}$.**Solution :**

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$\begin{aligned} F[e^{iax} f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx+iax} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx \end{aligned}$$

$$\boxed{F[e^{iax} f(x)] = F(s+a)}$$

5. State and prove the change of scale property of Fourier Transform.

Statement:

$$\text{If } F[f(x)] = F(s) \text{ then } F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx$$



Solution :

Given $F[f(x)] = F(s)$

The Fourier Transform of $f(x)$ is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx$$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx ,$$

If $a > 0$ Put $ax = t \Rightarrow adx = dt \Rightarrow dx = \frac{dt}{a}$ when $x = -\infty \Rightarrow t = -\infty$ and $x = \infty \Rightarrow t = \infty$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\left(\frac{s}{a}\right)t} \frac{dt}{a}$$

$$F[f(ax)] = \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\left(\frac{s}{a}\right)t} dt = \frac{1}{a} F\left(\frac{s}{a}\right). \text{---(1)}$$

If $a < 0$ Put $ax = t, adx = dt, dx = \frac{dt}{a}$

when $x = -\infty \Rightarrow t = \infty$ and $x = \infty \Rightarrow t = -\infty$

$$\Rightarrow F[f(ax)] = \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t) e^{i\left(\frac{s}{a}\right)t} \frac{dt}{a} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\left(\frac{s}{a}\right)t} \frac{dt}{a} = \frac{1}{a} F\left(\frac{s}{a}\right). \text{---(2)}$$

From (1) & (2) we get $F(f(ax)) = \frac{1}{|a|} F\left(\frac{s}{a}\right), a \neq 0$

6. Find the Fourier Sine transform of $\frac{1}{x}$.

Solution :

The Fourier Sine Transform of $f(x)$ is

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx dx$$

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin sx}{x} dx$$

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin t}{t} dt = \sqrt{\frac{2}{\pi}} \left(\frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}} \quad \because \int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}$$

PART-B

1. Find the Fourier transforms of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and hence evaluate $\int_0^{\infty} \frac{\sin x}{x} dx$. Using Parseval's

identity, prove that $\int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$.

Solution: Given $f(x) = \begin{cases} 1, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform $f(x)$ is



$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx.$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0 e^{isx} dx + \int_{-a}^a 1 e^{isx} dx + \int_a^{\infty} 0 e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a (\cos sx + i \sin sx) dx \quad \because e^{isx} = \cos sx + i \sin sx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^a \cos sx dx + i \int_{-a}^a \sin sx dx \right] \quad \because \sin sx \text{ is an odd fn} \Rightarrow \int_{-a}^a \sin sx dx = 0$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^a \cos sx dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}} \left[\frac{\sin sx}{s} \right]_0^a = \sqrt{\frac{2}{\pi}} \left[\frac{\sin as}{s} - 0 \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left[\frac{\sin as}{s} \right]$$

Deduction: 1

By inverse Fourier transform of $F(s)$.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{\sin as}{s} \right) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s} \right) (\cos sx - i \sin sx) ds$$

$$= \frac{\sqrt{2}}{\sqrt{2}\sqrt{\pi}\sqrt{\pi}} \left[\int_{-\infty}^{\infty} \left(\frac{\sin as}{s} \right) (\cos sx) ds - i \int_{-\infty}^{\infty} \left(\frac{\sin as}{s} \right) (\sin sx) ds \right]$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin as}{s} \right) \cos sx ds \quad \because \left(\frac{\sin as}{s} \right) \sin sx \text{ is an odd fn} \Rightarrow \int_{-\infty}^{\infty} \left(\frac{\sin as}{s} \right) \sin sx dx = 0$$

$$\int_0^{\infty} \left(\frac{\sin as}{s} \right) \cos sx ds = \frac{\pi}{2} f(x)$$

Put $x = 0$

$$\int_0^{\infty} \left(\frac{\sin as}{s} \right) \cos(0) ds = \frac{\pi}{2} f(0)$$

$$\int_0^{\infty} \left(\frac{\sin as}{s} \right) ds = \frac{\pi}{2} (1) \quad \because f(x) = 1 \Rightarrow f(0) = 1$$

Put $a=1$ and $s=x$ we get

$$\therefore \int_0^{\infty} \left(\frac{\sin x}{x} \right) dx = \frac{\pi}{2}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2 dx = \int_{-\infty}^{\infty} [f(x)]^2 ds$$



$$\int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}} \frac{\sin sa}{s} \right]^2 ds = \int_{-a}^a (1)^2 dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left[\frac{\sin sa}{s} \right]^2 ds = [x]_{-a}^a$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin sa}{s} \right)^2 ds = [a - (-a)]$$

$$\frac{2}{\pi} \cancel{\int_0^{\infty}} \left(\frac{\sin sa}{s} \right)^2 ds = \cancel{2} a$$

$$\int_0^{\infty} \left(\frac{\sin sa}{s} \right)^2 ds = \frac{\cancel{2} \pi a}{\cancel{4}} = \frac{\pi a}{2}$$

Put $a = 1$ & $s = t$ we get,

$$\boxed{\int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \int_0^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}}$$

2. Find the Fourier transform of $f(x) = \begin{cases} x; & \text{if } |x| < a \\ 0; & \text{if } |x| > a \end{cases}$.

Solution: Given $f(x) = \begin{cases} x, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform $f(x)$ is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0 e^{isx} dx + \int_{-a}^a x e^{isx} dx + \int_a^{\infty} 0 e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a x (\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^a x \cos sx dx + i \int_{-a}^a x \sin sx dx \right] \because x \cos sx \text{ is an odd fn } \therefore \int_{-a}^a x \cos sx dx = 0$$

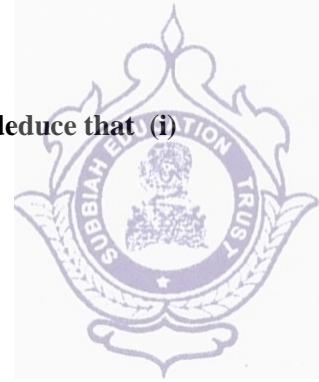
$$= i \frac{1}{\sqrt{2\pi}} 2 \int_0^a x \sin sx dx \because x \sin x \text{ is an even function } \Rightarrow \int_{-a}^a x \sin sx dx = 2 \int_0^a x \sin sx dx$$

$$= i \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}} \left[(x) \left(\frac{-\cos sx}{s} \right) - (1) \left(\frac{-\sin sx}{s^2} \right) \right]_0^a$$

$$= i \sqrt{\frac{2}{\pi}} \left[-\frac{x \cos sx}{s} + \frac{\sin sx}{s^2} \right]_0^a$$

$$= i \sqrt{\frac{2}{\pi}} \left[\left(-\frac{a \cos sa}{s} + \frac{\sin sa}{s^2} \right) - (0) \right]$$

$$\boxed{F(s) = i \sqrt{\frac{2}{\pi}} \left[\left(\frac{\sin sa - as \cos sa}{s^2} \right) \right]}$$



3. Find the Fourier transform of $f(x) = \begin{cases} a - |x|; & \text{if } |x| < a \\ 0; & \text{if } |x| > a > 0 \end{cases}$ is $\sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos as}{s^2} \right)$. Hence deduce that (i)

$$\int_0^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2} \quad \text{(ii)} \quad \int_0^{\infty} \left(\frac{\sin t}{t} \right)^4 dt = \frac{\pi}{3}$$

Solution: Given $f(x) = \begin{cases} a - |x|, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform $f(x)$ is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0 \cdot e^{isx} dx + \int_{-a}^a (a - |x|) e^{isx} dx + \int_a^{\infty} 0 \cdot e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a (a - |x|) (\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^a (a - |x|) \cos sx dx + i \int_{-a}^a (a - |x|) \sin sx dx \right]$$

$$\because (a - |x|) \sin sx \text{ is an odd fn} \Rightarrow \int_{-a}^a (a - |x|) \sin sx dx = 0$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^a (a - x) \cos sx dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}} \left[(a - x) \left(\frac{\sin sx}{s} \right) - (-1) \left(\frac{-\cos sx}{s^2} \right) \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos sx}{s^2} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(-\frac{1}{s^2} \right) (\cos sa - \cos 0) \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos sa}{s^2} \right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin^2 \left(\frac{as}{2} \right)}{s^2} \right]$$

$$\because \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \Rightarrow 1 - \cos \theta = 2 \sin^2 \left(\frac{\theta}{2} \right) \text{ here } \theta = \frac{as}{2}$$

Deduction: 1

By inverse Fourier transform of $F(s)$.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)e^{-isx} ds$$



$$\begin{aligned}
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(2\sqrt{\frac{2}{\pi}} \left[\frac{\sin^2\left(\frac{as}{2}\right)}{s^2} \right] \right) e^{-isx} ds \\
 &= \frac{2}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 (\cos sx - i \sin sx) ds \\
 &= \frac{2}{\pi} \left[\int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 (\cos sx) ds - i \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 (\sin sx) ds \right] \\
 f(x) &= \frac{4}{\pi} \left[\int_0^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 (\cos sx) ds \right] \quad \because \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 (\sin sx) \text{ is an odd function}
 \end{aligned}$$

$$\int_0^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 \cos sx ds = \frac{\pi}{4} f(x)$$

Put $x = 0$

$$\int_0^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 (\cos 0) ds = \frac{\pi}{4} f(0)$$

$$\int_0^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 ds = \frac{\pi a}{4} \quad \because f(x) = a - |x| \Rightarrow f(0) = a$$

Put $a=1$ and $s=t$ get

$$\int_0^{\infty} \left(\frac{\sin\left(\frac{s}{2}\right)}{s} \right)^2 ds = \frac{\pi}{4} \quad \text{put } \frac{s}{2} = t \Rightarrow \frac{ds}{2} = dt$$

$$\int_0^{\infty} \left(\frac{\sin t}{2t} \right)^2 2dt = \frac{\pi}{4}$$

$$\boxed{\therefore \int_0^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}}$$

(ii) By Parseval's identity,



$$\int_{-\infty}^{\infty} [F(s)]^2 dx = \int_{-\infty}^{\infty} [f(x)]^2 ds$$

$$\int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}} \left[\frac{\sin^2\left(\frac{as}{2}\right)}{s^2} \right] \right]^2 ds = \int_{-a}^a (a - |x|)^2 dx$$

$$\frac{8}{\pi} 2 \int_0^{\infty} \left[\frac{\sin\left(\frac{as}{2}\right)}{s} \right]^4 ds = 2 \int_0^a (a - x)^2 dx \quad \because (a - |x|)^2 \text{ and } \frac{\sin^2\left(\frac{as}{2}\right)}{s^2} \text{ are even functions}$$

$$\frac{8}{\pi} \int_0^{\infty} \left[\frac{\sin\left(\frac{as}{2}\right)}{s} \right]^4 ds = \int_0^a (a - x)^2 dx$$

$$\frac{8}{\pi} \int_0^{\infty} \left[\frac{\sin\left(\frac{as}{2}\right)}{s} \right]^4 ds = \left[\frac{(a - x)^3}{-3} \right]_0^a$$

$$\frac{8}{\pi} \int_0^{\infty} \left[\frac{\sin\left(\frac{as}{2}\right)}{s} \right]^4 ds = \left[(0) - \left(\frac{-a^3}{3} \right) \right]$$

$$\int_0^{\infty} \left[\frac{\sin\left(\frac{as}{2}\right)}{s} \right]^4 ds = \frac{a^3 \pi}{3 \times 8}$$

Put $a = 1$ & $s = t$ we get,

$$\int_0^{\infty} \left[\frac{\sin\left(\frac{s}{2}\right)}{s} \right]^4 ds = \frac{\pi}{24}$$

$$\text{put } \frac{s}{2} = t \Rightarrow \frac{ds}{2} = dt$$

$$\int_0^{\infty} \left[\frac{\sin t}{2t} \right]^4 2dt = \frac{\pi}{24}$$

$$\int_0^{\infty} \left[\frac{\sin t}{t} \right]^4 dt = \frac{\pi}{3}$$

4.

Find the Fourier transform of $f(x) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$ and hence find the value of



(i) $\int_0^{\infty} \frac{\sin^2 t}{t^2} dt$ (ii) $\int_0^{\infty} \frac{\sin^4 t}{t^4} dt$.

Solution:

Hint : In the previous problem $a=1$.

5.

Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| \leq a \\ 0, & |x| \geq a \end{cases}$ **and hence evaluate**

(i) $\int_0^{\infty} \left(\frac{\sin t - t \cos t}{t^3} \right) dt = \frac{\pi}{4}$ (ii) $\int_0^{\infty} \left(\frac{\sin t - t \cos t}{t^3} \right)^2 dt = \frac{\pi}{15}$

Solution: Given $f(x) = \begin{cases} a^2 - x^2, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform of $f(x)$ is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx.$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0 e^{isx} dx + \int_{-a}^a (a^2 - x^2) e^{isx} dx + \int_a^{\infty} 0 e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a (a^2 - x^2) (\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^a (a^2 - x^2) \cos sx dx + i \int_{-a}^a (a^2 - x^2) \sin sx dx \right]$$

$\therefore (a^2 - x^2) \sin sx$ is an *odd* fn. $\therefore \int_{-a}^a (a^2 - x^2) \sin sx dx = 0$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^a (a^2 - x^2) \cos sx dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}} \left[(a^2 - x^2) \left(\frac{\sin sx}{s} \right) - (-2x) \left(\frac{-\cos sx}{s^2} \right) + (-2) \left(\frac{-\sin sx}{s^3} \right) \right]_0^a$$

$$= -2\sqrt{\frac{2}{\pi}} \left[\frac{x \cos sx}{s^2} - \frac{\sin sx}{s^3} \right]_0^a$$

$$= -2\sqrt{\frac{2}{\pi}} \left[\left(\frac{a \cos sa}{s^2} - \frac{\sin sa}{s^3} \right) - (0) \right]$$

$$= -2\sqrt{\frac{2}{\pi}} \left[\frac{as \cos sa - \sin sa}{s^3} \right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin sa - as \cos sa}{s^3} \right]$$

Deduction: 1

By inverse Fourier transform of $F(s)$.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(2\sqrt{\frac{2}{\pi}} \left[\frac{\sin sa - as \cos sa}{s^3} \right] \right) e^{-isx} ds$$



$$= \frac{2}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \left[\frac{\sin sa - as \cos sa}{s^3} \right] (\cos sx - i \sin sx) ds$$

$$= \frac{2}{\pi} \left[\int_{-\infty}^{\infty} \left[\frac{\sin sa - as \cos sa}{s^3} \right] (\cos sx) ds - i \int_{-\infty}^{\infty} \left[\frac{\sin sa - as \cos sa}{s^3} \right] (\sin sx) ds \right]$$

$$f(x) = \frac{4}{\pi} \int_0^{\infty} \left[\frac{\sin sa - as \cos sa}{s^3} \right] \cos sx ds \quad \because \left[\frac{\sin sa - as \cos sa}{s^3} \right] (\sin sx) \text{ is an odd function}$$

$$\int_0^{\infty} \left[\frac{\sin sa - as \cos sa}{s^3} \right] \cos sx ds = \frac{\pi}{4} f(x)$$

Put x=0

$$\int_0^{\infty} \left[\frac{\sin sa - as \cos sa}{s^3} \right] (\cos 0) ds = \frac{\pi}{4} f(0)$$

$$\int_0^{\infty} \left[\frac{\sin sa - as \cos sa}{s^3} \right] ds = \frac{\pi a^2}{4} \quad \because f(x) = a^2 - x^2 \Rightarrow f(0) = a^2$$

Put a=1 and s=t get

$$\int_0^{\infty} \left[\frac{\sin t - t \cos t}{t^3} \right] dt = \frac{\pi}{4}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} [f(x)]^2 dx$$

$$\int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}} \left[\frac{\sin sa - as \cos sa}{s^3} \right] \right]^2 ds = \int_{-a}^a (a^2 - x^2)^2 dx$$

$$\frac{8}{\pi} \int_0^{\infty} \left[\frac{\sin sa - as \cos sa}{s^3} \right]^2 ds = 2 \int_0^a (a^4 - 2a^2x^2 + x^4) dx$$

$$\because (a^2 - x^2)^2 \text{ and } \left[\frac{\sin sa - as \cos sa}{s^3} \right]^2 \text{ are even functions}$$

$$\frac{8}{\pi} \int_0^{\infty} \left[\frac{\sin sa - as \cos sa}{s^3} \right]^2 ds = \left(a^4x - \frac{2a^2x^3}{3} + \frac{x^5}{5} \right)_0^a$$

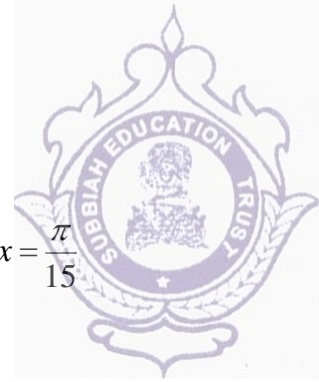
$$\frac{8}{\pi} \int_0^{\infty} \left[\frac{\sin sa - as \cos sa}{s^3} \right]^2 ds = \left(a^5 - \frac{2a^5}{3} + \frac{a^5}{5} \right)$$

$$\frac{8}{\pi} \int_0^{\infty} \left[\frac{\sin sa - as \cos sa}{s^3} \right]^2 ds = \left(\frac{15a^5 - 10a^5 + 3a^5}{15} \right)$$

$$\int_0^{\infty} \left[\frac{\sin sa - as \cos sa}{s^3} \right]^2 ds = \left(\frac{8a^5}{15} \right) \times \frac{\pi}{8}$$

Put a=1 & s=t we get,

$$\int_0^{\infty} \left[\frac{\sin t - t \cos t}{t^3} \right]^2 dt = \frac{\pi}{15}$$



6. Find the Fourier transform of $f(x) = \begin{cases} 1-x^2; & \text{if } |x| < 1 \\ 0; & \text{if } |x| \geq 1 \end{cases}$.

Hence show that (i) $\int_0^\infty \left[\frac{\sin s - s \cos s}{s^2} \right] \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$ and (ii) $\int_0^\infty \frac{(x \cos x - \sin x)^2}{x^6} dx = \frac{\pi}{15}$

Solution: Given $f(x) = \begin{cases} 1-x^2, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform $f(x)$ is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx.$$

$$\begin{aligned} F(s) &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} 0 e^{isx} dx + \int_{-1}^1 (1-x^2) e^{isx} dx + \int_1^{\infty} 0 e^{isx} dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 (1-x^2) (\cos sx + i \sin sx) dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^1 (1-x^2) \cos sx dx + i \int_{-1}^1 (1-x^2) \sin sx dx \right] \end{aligned}$$

$$\because (1-x^2) \sin sx \text{ is an odd fn. } \therefore \int_{-1}^1 (1-x^2) \sin sx dx = 0$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^1 (1-x^2) \cos sx dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}} \left[(1-x^2) \left(\frac{\sin sx}{s} \right) - (-2x) \left(\frac{-\cos sx}{s^2} \right) + (-2) \left(\frac{-\sin sx}{s^3} \right) \right]_0^1$$

$$= -2\sqrt{\frac{2}{\pi}} \left[\frac{x \cos sx}{s^2} - \frac{\sin sx}{s^3} \right]_0^1$$

$$= -2\sqrt{\frac{2}{\pi}} \left[\left(\frac{\cos s}{s^2} - \frac{\sin s}{s^3} \right) - (0) \right]$$

$$= -2\sqrt{\frac{2}{\pi}} \left[\frac{s \cos s - \sin s}{s^3} \right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s \cos s}{s^3} \right]$$

Deduction: 1

By inverse Fourier transform of $F(s)$.

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)e^{-isx} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s \cos s}{s^3} \right] \right) e^{-isx} ds \\ &= \frac{2}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \left[\frac{\sin s - s \cos s}{s^3} \right] (\cos sx - i \sin sx) ds \end{aligned}$$



$$= \frac{2}{\pi} \left[\int_{-\infty}^{\infty} \left[\frac{\sin s - s \cos s}{s^3} \right] (\cos sx) ds - i \int_{-\infty}^{\infty} \left[\frac{\sin s - s \cos s}{s^3} \right] (\sin sx) ds \right]$$

$$f(x) = \frac{4}{\pi} \int_0^{\infty} \left[\frac{\sin s - s \cos s}{s^3} \right] \cos sx ds \quad \because \left[\frac{\sin s - s \cos s}{s^3} \right] (\sin sx) \text{ is an odd function}$$

$$\int_0^{\infty} \left[\frac{\sin s - s \cos s}{s^3} \right] \cos sx ds = \frac{\pi}{4} f(x)$$

Put $x = \frac{1}{2}$

$$\int_0^{\infty} \left[\frac{\sin s - s \cos s}{s^3} \right] \cos \left(\frac{s}{2} \right) ds = \frac{\pi}{4} f \left(\frac{1}{2} \right)$$

$$\int_0^{\infty} \left[\frac{\sin s - s \cos s}{s^3} \right] \cos \left(\frac{s}{2} \right) ds = \frac{\pi}{4} \times \frac{3}{4} \quad \because f(x) = 1 - x^2 \Rightarrow f \left(\frac{1}{2} \right) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\boxed{\int_0^{\infty} \left[\frac{\sin s - s \cos s}{s^3} \right] \cos \left(\frac{s}{2} \right) ds = \frac{3\pi}{16}}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} [f(x)]^2 dx$$

$$\int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s \cos s}{s^3} \right] \right]^2 ds = \int_{-1}^1 (1 - x^2)^2 dx$$

$$\frac{8}{\pi} \int_0^{\infty} \left[\frac{\sin sa - as \cos sa}{s^3} \right]^2 ds = 2 \int_0^1 (1 - 2x^2 + x^4) dx$$

$\because (1 - x^2)^2$ and $\left[\frac{\sin s - s \cos s}{s^3} \right]^2$ are an even functions

$$\frac{8}{\pi} \int_0^{\infty} \left[\frac{\sin s - s \cos s}{s^3} \right]^2 ds = \left(x - \frac{2x^3}{3} + \frac{x^5}{5} \right)_0^1$$

$$\frac{8}{\pi} \int_0^{\infty} \left[\frac{\sin s - s \cos s}{s^3} \right]^2 ds = \left(1 - \frac{2}{3} + \frac{1}{5} \right)$$

$$\frac{8}{\pi} \int_0^{\infty} \left[\frac{\sin s - s \cos s}{s^3} \right]^2 ds = \left(\frac{15 - 10 + 3}{15} \right)$$

$$\int_0^{\infty} \left[\frac{\sin s - s \cos s}{s^3} \right]^2 ds = \left(\frac{8}{15} \right) \times \frac{\pi}{8}$$

Put $s=t$ we get,

$$\boxed{\int_0^{\infty} \frac{(\sin t - t \cos t)^2}{t^6} dx = \frac{\pi}{15}}$$

4.2 FOURIER SINE AND COSINE TRANSFORMS



Find the Fourier cosine and sine transform of
Solution: $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2-x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$

Given $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2-x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$

The Fourier Cosine transform of f(x) is

$$F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx .$$

$$\begin{aligned} &= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos sx \, dx + \int_1^2 (2-x) \cos sx \, dx + \int_2^{\infty} 0 \cos sx \, dx \right] \\ &= \sqrt{\frac{2}{\pi}} \left\{ \left[(x) \left(\frac{\sin sx}{s} \right) - (1) \left(\frac{-\cos sx}{s^2} \right) \right]_0^1 + \left[(2-x) \left(\frac{\sin sx}{s} \right) - (-1) \left(\frac{-\cos sx}{s^2} \right) \right]_1^2 \right\} \\ &= \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{x \sin sx}{s} + \frac{\cos sx}{s^2} \right]_0^1 + \left[(2-x) \left(\frac{\sin sx}{s} \right) - \frac{\cos sx}{s^2} \right]_1^2 \right\} \\ &= \sqrt{\frac{2}{\pi}} \left\{ \left[\left(\frac{\sin s}{s} + \frac{\cos s}{s^2} \right) - \left(0 + \frac{1}{s^2} \right) \right] + \left[\left(0 - \frac{\cos 2s}{s^2} \right) - \left(\frac{\sin s}{s} - \frac{\cos s}{s^2} \right) \right] \right\} \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2} \right] \end{aligned}$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \left[\frac{2 \cos s - \cos 2s - 1}{s^2} \right]$$

The Fourier sine transform of f(x) is

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx .$$

$$\begin{aligned} &= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin sx \, dx + \int_1^2 (2-x) \sin sx \, dx + \int_2^{\infty} 0 \sin sx \, dx \right] \\ &= \sqrt{\frac{2}{\pi}} \left\{ \left[(x) \left(\frac{-\cos sx}{s} \right) - (1) \left(\frac{-\sin sx}{s^2} \right) \right]_0^1 + \left[(2-x) \left(\frac{-\cos sx}{s} \right) - (-1) \left(\frac{-\sin sx}{s^2} \right) \right]_1^2 \right\} \\ &= \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{-x \cos sx}{s} + \frac{\sin sx}{s^2} \right]_0^1 + \left[-(2-x) \left(\frac{\cos sx}{s} \right) - \frac{\sin sx}{s^2} \right]_1^2 \right\} \\ &= \sqrt{\frac{2}{\pi}} \left\{ \left[\left(-\frac{\cos s}{s} + \frac{\sin s}{s^2} \right) - (0) \right] + \left[\left(0 - \frac{\sin 2s}{s^2} \right) - \left(-\frac{\cos s}{s} - \frac{\sin s}{s^2} \right) \right] \right\} \\ &= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos s}{s} + \frac{\sin s}{s^2} - \frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2} \right] \end{aligned}$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \left[\frac{2 \sin s - \sin 2s}{s^2} \right]$$



Find Fourier transform of $e^{-a|x|}$ and hence deduce that

(a) $\int_0^{\infty} \frac{\cos xt}{a^2 + t^2} dt = \frac{\pi}{2a} e^{-a|x|}$ (b) $F[xe^{-a|x|}] = i\sqrt{\frac{2}{\pi}} \frac{2as}{(s^2 + a^2)^2}$.

The Fourier transform of $f(x)$ is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{-a|x|} \cos sx dx + i \int_{-\infty}^{\infty} e^{-a|x|} \sin sx dx \right]$$

$\therefore e^{-a|x|} \sin sx$ is an odd fn. $\therefore \int_{-\infty}^{\infty} e^{-a|x|} \sin sx dx = 0$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^{\infty} e^{-ax} \cos sx dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}} \int_0^{\infty} e^{-ax} \cos sx dx$$

$$F(s) = F[e^{-a|x|}] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \quad \therefore \int_0^{\infty} e^{-ax} \cos bx dx = \frac{a}{a^2 + b^2} \text{ here } a = a; b = s$$

Deduction (a):

By inverse Fourier transform of $F(s)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \left[\frac{a}{a^2 + s^2} \right] (\cos sx - i \sin sx) ds$$

$$= \frac{a}{\pi} \left[\int_{-\infty}^{\infty} \left[\frac{1}{a^2 + s^2} \right] (\cos sx) ds - ia \int_{-\infty}^{\infty} \left[\frac{1}{a^2 + s^2} \right] (\sin sx) ds \right]$$

$$f(x) = \frac{2a}{\pi} \int_0^{\infty} \left[\frac{1}{a^2 + s^2} \right] \cos sx ds \quad \therefore \left(\frac{1}{a^2 + s^2} \right) (\sin sx) \text{ is an odd function}$$

$$\int_0^{\infty} \left(\frac{1}{a^2 + s^2} \right) \cos sx ds = \frac{\pi}{2a} f(x)$$

$$\int_0^{\infty} \frac{\cos sx}{a^2 + s^2} ds = \frac{\pi}{2a} e^{-a|x|}$$

Put $s=t$

$$\int_0^{\infty} \frac{\cos tx}{t^2 + a^2} dt = \frac{\pi}{2a} e^{-a|x|}$$

Deduction (b):

By Property



$$F[x f(x)] = -i \frac{d}{ds} [F(s)]$$

$$F[x e^{-a|x|}] = -i \frac{d}{ds} F(e^{-a|x|})$$

$$= -i \frac{d}{ds} \left(\sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2} \right)$$

$$= -ia \sqrt{\frac{2}{\pi}} \left(\frac{-1}{(a^2 + s^2)^2} (0 + 2s) \right) = i \sqrt{\frac{2}{\pi}} \frac{2as}{(a^2 + s^2)^2}$$

$$F[x e^{-a|x|}] = i \sqrt{\frac{2}{\pi}} \left(\frac{2as}{(s^2 + a^2)^2} \right)$$

Find the Fourier sine and cosine transform of e^{-ax} , $a > 0$ and deduce that

i) $\int_0^{\infty} \frac{s}{s^2 + a^2} \sin sx \, dx = \frac{\pi}{2} e^{-ax}$.

ii) $\int_0^{\infty} \frac{1}{s^2 + a^2} \cos sx \, dx = \frac{\pi}{2a} e^{-ax}$

Solution:

The Fourier sine transform of $f(x)$ is

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx$$

$$F_s(s) = F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right]$$

$$\because \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \text{ here } a = a; b = s$$

The Fourier cosine transform of $f(x)$ is

$$F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx \, dx$$

$$F_c(s) = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right]$$

$$\because \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \text{ here } a = a; b = s$$

The inverse Fourier sine transform of $F_s(s)$ is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right] \sin sx \, dx$$

$$= \frac{2}{\pi} \int_0^{\infty} \left[\frac{s}{a^2 + s^2} \right] \sin sx \, dx$$



$$\int_0^{\infty} \left[\frac{s}{a^2 + s^2} \right] \sin sx \, dx = \frac{\pi}{2} f(x)$$

$$\boxed{\int_0^{\infty} \left[\frac{s}{a^2 + s^2} \right] \sin sx \, dx = \frac{\pi}{2} e^{-ax}}$$

The inverse Fourier Cosine transform of $F_c(s)$ is

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \cos sx \, dx \\ &= \frac{2a}{\pi} \int_0^{\infty} \left[\frac{1}{a^2 + s^2} \right] \cos sx \, dx \end{aligned}$$

$$\int_0^{\infty} \left[\frac{a}{a^2 + s^2} \right] \cos sx \, dx = \frac{\pi}{2} f(x)$$

$$\boxed{\int_0^{\infty} \left[\frac{a}{a^2 + s^2} \right] \cos sx \, dx = \frac{\pi}{2a} e^{-ax}}$$

Find the Fourier sine and cosine transform of e^{-ax} , $a > 0$ and hence find $F_c[xe^{-ax}]$ and $F_s[xe^{-ax}]$.

Solution:

The Fourier sine transform $f(x)$ is

$$\begin{aligned} F_s[f(x)] &= F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx \end{aligned}$$

$$\boxed{F_s(s) = F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right]}$$

$$\therefore \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \quad \text{here } a = a; b = s$$

The Fourier cosine transform $f(x)$ is

$$\begin{aligned} F_c[f(x)] &= F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx \, dx \end{aligned}$$

$$\boxed{F_c(s) = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right]}$$

$$\therefore \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \quad \text{here } a = a; b = s$$

We know that

$$\text{i) } F_s[xf(x)] = -\frac{d}{ds} \{F_c[f(x)]\} = -\frac{d}{ds} [F_c(s)]$$

$$\begin{aligned} F_s[xe^{-ax}] &= -\frac{d}{ds} \{F_c[e^{-ax}]\} = -\frac{d}{ds} \left\{ \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \right\} \\ &= -a \sqrt{\frac{2}{\pi}} \frac{d}{ds} \left\{ \frac{1}{a^2 + s^2} \right\} \end{aligned}$$



$$= -a\sqrt{\frac{2}{\pi}} \left[\frac{-1}{(a^2 + s^2)^2} (0 + 2s) \right]$$

$$F_s [xe^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{2as}{(a^2 + s^2)^2} \right]$$

ii) $F_c [xf(x)] = \frac{d}{ds} \{F_s [f(x)]\} = \frac{d}{ds} [F_s(s)]$

$$F_s [xe^{-ax}] = \frac{d}{ds} \{F_c [e^{-ax}]\} = \frac{d}{ds} \left\{ \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right] \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{(a^2 + s^2)(1) - s(0 + 2s)}{(a^2 + s^2)^2} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{a^2 + s^2 - 2s^2}{(a^2 + s^2)^2} \right\}$$

$$F_s [xe^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{a^2 - s^2}{(a^2 + s^2)^2} \right]$$

Find the Fourier sine transform of $\frac{e^{-ax}}{x}, a > 0$ and hence find $F_s \left[\frac{e^{-ax} - e^{-bx}}{x} \right]$.

Solution:

The Fourier sine transform of f(x) is

$$F_s [f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$F_s \left[\frac{e^{-ax}}{x} \right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx$$

Taking diff. on both sides w.r.to s

$$\frac{d}{ds} \left\{ F_s \left[\frac{e^{-ax}}{x} \right] \right\} = \frac{d}{ds} \left\{ \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx \right\}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \frac{\partial}{\partial s} (\sin sx) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} (\cos sx) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$\frac{d}{ds} \left\{ F_s \left[\frac{e^{-ax}}{x} \right] \right\} = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right]$$

Integrating on on both sides w.r.to s



$$F_s \left[\frac{e^{-ax}}{x} \right] = \sqrt{\frac{2}{\pi}} \int \left[\frac{a}{a^2 + s^2} \right] ds$$

$$\boxed{F_s \left[\frac{e^{-ax}}{x} \right] = \sqrt{\frac{2}{\pi}} \tan^{-1} \left(\frac{s}{a} \right)} \quad \because \int \frac{a}{x^2 + a^2} dx = \tan^{-1} \left(\frac{x}{a} \right)$$

Similarly, $F_s \left[\frac{e^{-bx}}{x} \right] = \sqrt{\frac{2}{\pi}} \tan^{-1} \left(\frac{s}{b} \right)$

Deduction:

$$\begin{aligned} F_s \left[\frac{e^{-ax} - e^{-bx}}{x} \right] &= F_s \left[\frac{e^{-ax}}{x} - \frac{e^{-bx}}{x} \right] \\ &= F_s \left[\frac{e^{-ax}}{x} \right] - F_s \left[\frac{e^{-bx}}{x} \right] \\ &= \sqrt{\frac{2}{\pi}} \tan^{-1} \left(\frac{s}{a} \right) - \sqrt{\frac{2}{\pi}} \tan^{-1} \left(\frac{s}{b} \right) \end{aligned}$$

$$\boxed{F_s \left[\frac{e^{-ax} - e^{-bx}}{x} \right] = \sqrt{\frac{2}{\pi}} \left[\tan^{-1} \left(\frac{s}{a} \right) - \tan^{-1} \left(\frac{s}{b} \right) \right]}$$

Find the Fourier cosine transform of $\frac{e^{-ax}}{x}, a > 0$ and hence find $F_c \left[\frac{e^{-ax} - e^{-bx}}{x} \right]$

Solution:

The Fourier cosine transform f(x) is

$$F_c [f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$$

$$F_c \left[\frac{e^{-ax}}{x} \right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \cos sx \, dx$$

Taking diff. on both sides w.r.to s

$$\begin{aligned} \frac{d}{ds} \left\{ F_c \left[\frac{e^{-ax}}{x} \right] \right\} &= \frac{d}{ds} \left\{ \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \cos sx \, dx \right\} \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \frac{\partial}{\partial s} (\cos sx) \, dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} (-\sin sx) \, dx \\ &= -\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx \end{aligned}$$

$$\frac{d}{ds} \left\{ F_c \left[\frac{e^{-ax}}{x} \right] \right\} = -\sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right]$$

Integrating on on both sides w.r.to s

$$\begin{aligned} F_c \left[\frac{e^{-ax}}{x} \right] &= -\sqrt{\frac{2}{\pi}} \int \left[\frac{s}{a^2 + s^2} \right] ds \\ &= -\sqrt{\frac{2}{\pi}} \int \left[\frac{s}{a^2 + s^2} \right] ds \end{aligned}$$



$$\begin{aligned}
 &= -\sqrt{\frac{2}{\pi}} \frac{1}{2} \int \left[\frac{2s}{a^2 + s^2} \right] ds \\
 &= -\sqrt{\frac{2}{\pi}} \frac{1}{2} \log(s^2 + a^2) \quad \because \int \frac{f'(x)}{f(x)} dx = \log[f(x)] \\
 &= -\frac{1}{\sqrt{2\pi}} \log(s^2 + a^2) \\
 &= \frac{1}{\sqrt{2\pi}} \log\left(\frac{1}{s^2 + a^2}\right)
 \end{aligned}$$

$$\boxed{F_c \left[\frac{e^{-ax}}{x} \right] = \frac{1}{\sqrt{2\pi}} \log\left(\frac{1}{s^2 + a^2}\right)}$$

Similarly $F_c \left[\frac{e^{-bx}}{x} \right] = \frac{1}{\sqrt{2\pi}} \log\left(\frac{1}{s^2 + b^2}\right)$

Deduction:

$$\begin{aligned}
 F_c \left[\frac{e^{-ax} - e^{-bx}}{x} \right] &= F_c \left[\frac{e^{-ax}}{x} - \frac{e^{-bx}}{x} \right] \\
 &= F_c \left[\frac{e^{-ax}}{x} \right] - F_c \left[\frac{e^{-bx}}{x} \right] \\
 &= \frac{1}{\sqrt{2\pi}} \log\left(\frac{1}{s^2 + a^2}\right) - \frac{1}{\sqrt{2\pi}} \log\left(\frac{1}{s^2 + b^2}\right) \\
 &= \frac{1}{\sqrt{2\pi}} \log\left(\frac{s^2 + b^2}{s^2 + a^2}\right)
 \end{aligned}$$

$$\boxed{F_s \left[\frac{e^{-ax} - e^{-bx}}{x} \right] = \frac{1}{\sqrt{2\pi}} \log\left(\frac{s^2 + b^2}{s^2 + a^2}\right)}$$

13. Using Parseval's identity evaluate the following integrals.

1) $\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2}$ 2) $\int_0^{\infty} \frac{x^2}{(x^2 + a^2)^2} dx$, where $a > 0$.

Solution:

Assume $f(x) = e^{-ax}$

The Fourier sine transform $f(x)$ is

$$\begin{aligned}
 F_s[f(x)] = F_s(s) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx \\
 &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx
 \end{aligned}$$

$$\boxed{F_s(s) = F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right]} \quad \because \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \text{ here } a = a; b = s$$

The Fourier cosine transform $f(x)$ is

$$\begin{aligned}
 F_c[f(x)] = F_c(s) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx \\
 &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx \, dx
 \end{aligned}$$



$$F_c(s) = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right]$$

$$\therefore \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \text{ here } a = a; b = s$$

(i) The Parseval's identity for Fourier cosine transform is

$$\int_0^{\infty} |F_c(s)|^2 \, ds = \int_0^{\infty} |f(x)|^2 \, dx$$

$$\int_0^{\infty} \left(\sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \right)^2 \, ds = \int_0^{\infty} (e^{-ax})^2 \, dx$$

$$\frac{2a^2}{\pi} \int_0^{\infty} \frac{1}{(a^2 + s^2)^2} \, ds = \int_0^{\infty} e^{-2ax} \, dx$$

$$\int_0^{\infty} \frac{1}{(a^2 + s^2)^2} \, ds = \frac{\pi}{2a^2} \left[\frac{e^{-2ax}}{-2a} \right]_0^{\infty}$$

$$\int_0^{\infty} \frac{1}{(a^2 + s^2)^2} \, ds = \frac{-\pi}{4a^3} [e^{-\infty} - e^{-0}]$$

$$\int_0^{\infty} \frac{1}{(a^2 + s^2)^2} \, ds = \frac{-\pi}{4a^3} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^{\infty} \frac{1}{(a^2 + s^2)^2} \, ds = \frac{\pi}{4a^3}$$

Put s=x we get

$$\int_0^{\infty} \frac{1}{(a^2 + x^2)^2} \, dx = \frac{\pi}{4a^3}$$

(ii) The Parseval's identity for Fourier sine transform is

$$\int_0^{\infty} |F_s(s)|^2 \, ds = \int_0^{\infty} |f(x)|^2 \, dx$$

$$\int_0^{\infty} \left(\sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right] \right)^2 \, ds = \int_0^{\infty} (e^{-ax})^2 \, dx$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{s^2}{(a^2 + s^2)^2} \, ds = \int_0^{\infty} e^{-2ax} \, dx$$

$$\int_0^{\infty} \frac{s^2}{(a^2 + s^2)^2} \, ds = \frac{\pi}{2} \left[\frac{e^{-2ax}}{-2a} \right]_0^{\infty}$$

$$\int_0^{\infty} \frac{s^2}{(a^2 + s^2)^2} \, ds = \frac{-\pi}{4a} [e^{-\infty} - e^{-0}]$$

$$\int_0^{\infty} \frac{s^2}{(a^2 + s^2)^2} \, ds = \frac{-\pi}{4a} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^{\infty} \frac{s^2}{(a^2 + s^2)^2} \, ds = \frac{\pi}{4a}$$

$$\int_0^{\infty} \frac{x^2}{(a^2 + x^2)^2} \, dx = \frac{\pi}{4a}$$



4.3 CONVOLUTION OF FOURIER TRANSFORM

Evaluate (a) $\int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$ **(b)** $\int_0^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)} dx$ **using Fourier transforms.**

Solution:

(a) Assume $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

The Fourier sine transform f(x) is

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx .$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx$$

$$F_s(s) = F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right]$$

$$\because \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \text{ here } a = a; b = s$$

Similarly

$$G_s(s) = F_s[e^{-bx}] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{b^2 + s^2} \right]$$

We know that

$$\int_0^{\infty} F_s(s)G_s(s)ds = \int_0^{\infty} f(x)g(x)dx$$

$$\int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right] \sqrt{\frac{2}{\pi}} \left[\frac{s}{b^2 + s^2} \right] ds = \int_0^{\infty} e^{-ax} e^{-bx} dx$$

$$\frac{2}{\pi} \int_0^{\infty} \left[\frac{s^2}{(a^2 + s^2)(b^2 + s^2)} \right] ds = \int_0^{\infty} e^{-ax-bx} dx$$

$$\int_0^{\infty} \left[\frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right] ds = \frac{\pi}{2} \int_0^{\infty} e^{-(a+b)x} dx$$

$$= \frac{\pi}{2} \int_0^{\infty} e^{-(a+b)x} dx$$

$$= \frac{\pi}{2} \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^{\infty}$$

$$= \frac{-\pi}{2(a+b)} [e^{-\infty} - e^{-0}]_0^{\infty}$$

$$= \frac{-\pi}{2(a+b)} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^{\infty} \left[\frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right] ds = \frac{\pi}{2(a+b)}$$

Put s=x we get



$$\int_0^{\infty} \left[\frac{x^2}{(x^2 + a^2)(x^2 + b^2)} \right] dx = \frac{\pi}{2(a+b)}$$

(b) Assume $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

The Fourier cosine transform f(x) is

$$\begin{aligned} F_c[f(x)] &= F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx . \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx \, dx \end{aligned}$$

$$F_c(s) = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \quad \because \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \text{ here } a = a; b = s$$

Similarly

$$G_c(s) = F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}} \left[\frac{b}{b^2 + s^2} \right]$$

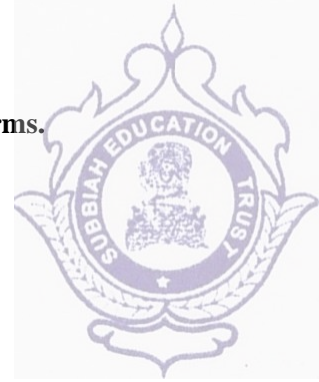
We know that

$$\begin{aligned} \int_0^{\infty} F_c(s)G_c(s)ds &= \int_0^{\infty} f(x)g(x)dx \\ \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \sqrt{\frac{2}{\pi}} \left[\frac{b}{b^2 + s^2} \right] ds &= \int_0^{\infty} e^{-ax} e^{-bx} dx \\ \frac{2ab}{\pi} \int_0^{\infty} \left[\frac{1}{(a^2 + s^2)(b^2 + s^2)} \right] ds &= \int_0^{\infty} e^{-ax-bx} dx \\ \int_0^{\infty} \left[\frac{1}{(s^2 + a^2)(s^2 + b^2)} \right] ds &= \frac{\pi}{2ab} \int_0^{\infty} e^{-(a+b)x} dx \\ &= \frac{\pi}{2ab} \int_0^{\infty} e^{-(a+b)x} dx \\ &= \frac{\pi}{2ab} \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^{\infty} \\ &= \frac{-\pi}{2ab(a+b)} \left[e^{-\infty} - e^{-0} \right]_0^{\infty} \\ &= \frac{-\pi}{2ab(a+b)} [0-1] \quad \because e^{-\infty} = 0; e^{-0} = 1 \end{aligned}$$

$$\int_0^{\infty} \left[\frac{1}{(s^2 + a^2)(s^2 + b^2)} \right] ds = \frac{\pi}{2ab(a+b)}$$

Put s=x we get

$$\int_0^{\infty} \left[\frac{1}{(x^2 + a^2)(x^2 + b^2)} \right] dx = \frac{\pi}{2ab(a+b)}$$



Evaluate (a) $\int_0^\infty \frac{x^2}{(x^2+9)(x^2+16)} dx$, (b) $\int_0^\infty \frac{1}{(x^2+1)(x^2+4)} dx$ using Fourier transforms.

Solution:

(a) Assume $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

The Fourier sine transform f(x) is

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx dx .$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx dx$$

$$F_s(s) = F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right]$$

$$\therefore \int_0^\infty e^{-ax} \sin bx dx = \frac{b}{a^2 + b^2} \text{ here } a = a; b = s$$

Similarly

$$G_s(s) = G_s[e^{-bx}] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{b^2 + s^2} \right]$$

We know that

$$\int_0^\infty F_s(s)G_s(s)ds = \int_0^\infty f(x)g(x)dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right] \sqrt{\frac{2}{\pi}} \left[\frac{s}{b^2 + s^2} \right] ds = \int_0^\infty e^{-ax} e^{-bx} dx$$

$$\frac{2}{\pi} \int_0^\infty \left[\frac{s^2}{(a^2 + s^2)(b^2 + s^2)} \right] ds = \int_0^\infty e^{-ax-bx} dx$$

$$\int_0^\infty \left[\frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right] ds = \frac{\pi}{2} \int_0^\infty e^{-(a+b)x} dx$$

$$= \frac{\pi}{2} \int_0^\infty e^{-(a+b)x} dx$$

$$= \frac{\pi}{2} \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^\infty$$

$$= \frac{-\pi}{2(a+b)} [e^{-\infty} - e^{-0}]_0^\infty$$

$$= \frac{-\pi}{2(a+b)} [0-1] \quad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^\infty \left[\frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right] ds = \frac{\pi}{2(a+b)} \text{-----(1)}$$

Put a=3 & b=4 and s=x we get

$$(1) \Rightarrow \int_0^\infty \frac{x^2}{(x^2+9)(x^2+16)} dx = \frac{\pi}{2(3+4)}$$



$$\int_0^{\infty} \frac{x^2}{(x^2 + 9)(x^2 + 16)} dx = \frac{\pi}{14}$$

(b) Assume $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

The Fourier cosine transform f(x) is

$$\begin{aligned} F_c[f(x)] &= F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx dx . \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx dx \end{aligned}$$

$$F_c(s) = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \quad \because \int_0^{\infty} e^{-ax} \cos bx dx = \frac{a}{a^2 + b^2} \text{ here } a = a; b = s$$

Similarly

$$G_c(s) = F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}} \left[\frac{b}{b^2 + s^2} \right]$$

We know that

$$\begin{aligned} \int_0^{\infty} F_c(s)G_c(s)ds &= \int_0^{\infty} f(x)g(x)dx \\ \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \sqrt{\frac{2}{\pi}} \left[\frac{b}{b^2 + s^2} \right] ds &= \int_0^{\infty} e^{-ax} e^{-bx} dx \\ \frac{2ab}{\pi} \int_0^{\infty} \left[\frac{1}{(a^2 + s^2)(b^2 + s^2)} \right] ds &= \int_0^{\infty} e^{-ax-bx} dx \\ \int_0^{\infty} \left[\frac{1}{(s^2 + a^2)(s^2 + b^2)} \right] ds &= \frac{\pi}{2ab} \int_0^{\infty} e^{-(a+b)x} dx \\ &= \frac{\pi}{2ab} \int_0^{\infty} e^{-(a+b)x} dx \\ &= \frac{\pi}{2ab} \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^{\infty} \\ &= \frac{-\pi}{2ab(a+b)} \left[e^{-\infty} - e^{-0} \right]_0^{\infty} \\ &= \frac{-\pi}{2ab(a+b)} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1 \end{aligned}$$

$$\int_0^{\infty} \left[\frac{1}{(s^2 + a^2)(s^2 + b^2)} \right] ds = \frac{\pi}{2ab(a+b)}$$

Put a=1 & b=2 s=x we get

$$(1) \Rightarrow \int_0^{\infty} \frac{1}{(x^2 + 1)(x^2 + 4)} dx = \frac{\pi}{2(1)(2)(1+2)}$$

$$\int_0^{\infty} \frac{1}{(x^2 + 1)(x^2 + 4)} dx = \frac{\pi}{12}$$

4.4 SELF RECIPROCAL UNDER FOURIER TRANSFORM



Self-reciprocal:

If a transformation of a function $f(x)$ is equal to $f(s)$ then the function $f(x)$ is called **self-reciprocal**.

Find the Fourier transform of $e^{-a^2x^2}$. Hence prove that $e^{-\frac{x^2}{2}}$ is self-reciprocal with respect to Fourier Transforms.

Solution:

The Fourier transform $f(x)$ is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx$$

$$\begin{aligned} F[e^{-a^2x^2}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2 + isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[(ax)^2 - isx + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2 \right]} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{\left(\frac{is}{2a}\right)^2} dx \\ &= \frac{1}{\sqrt{2\pi}} e^{\left(\frac{is}{2a}\right)^2} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx \end{aligned}$$

$$\begin{aligned} (A - B)^2 &= A^2 - 2AB + B^2 \\ 2AB &= isx \\ \text{Here } A &= ax, B = \frac{is}{2a} \end{aligned}$$

Let $u = ax - \frac{is}{2a} \Rightarrow du = adx \Rightarrow dx = \frac{du}{a}$; $u : -\infty$ to ∞

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}} e^{\frac{i^2s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a} \\ &= \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} du \quad \because i^2 = -1 \\ &= \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} 2 \int_0^{\infty} e^{-u^2} du \quad \because e^{-u^2} \text{ is an even function} \\ &= \frac{1}{a\sqrt{2}\sqrt{\pi}} e^{\frac{-s^2}{4a^2}} 2 \frac{\sqrt{\pi}}{2} \quad \because \int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2} \end{aligned}$$

$$F[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}} \quad \text{-----(1)}$$

Deduction:

To prove $e^{-\frac{x^2}{2}}$ is self-reciprocal

It is enough to prove that $F\left[e^{-\frac{x^2}{2}}\right]$ is $e^{-\frac{s^2}{2}}$



Put $a = \frac{1}{\sqrt{2}}$ in (1)

$$F \left[e^{-\left(\frac{1}{\sqrt{2}}\right)^2 x^2} \right] = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\sqrt{2}} e^{\frac{-s^2}{4\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$F \left[e^{-\frac{x^2}{2}} \right] = e^{\frac{-s^2}{2}}$$

$$F \left[e^{-\frac{x^2}{2}} \right] = e^{\frac{-s^2}{2}}$$

$\therefore e^{\frac{-x^2}{2}}$ is self reciprocal.

Find the Fourier transform of $e^{\frac{-x^2}{2}}$.

(or) Show that $e^{\frac{-x^2}{2}}$ is self-reciprocal with respect to Fourier Transforms.

Solution:

Let $f(x) = e^{\frac{-x^2}{2}}$

Assume $f(x) = e^{-a^2 x^2}$ where $a = \frac{1}{\sqrt{2}}$

The Fourier transform f(x) is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx$$

$$\begin{aligned} F[e^{-a^2 x^2}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2 x^2} e^{isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2 x^2 + isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2 x^2 - isx)} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left[(ax)^2 - isx + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2 \right]} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{\left(\frac{is}{2a}\right)^2} dx \\ &= \frac{1}{\sqrt{2\pi}} e^{\left(\frac{is}{2a}\right)^2} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx \end{aligned}$$

Let $u = ax - \frac{is}{2a} \Rightarrow du = adx \Rightarrow dx = \frac{du}{a}$; $u : -\infty$ to ∞

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{i^2 s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a}$$

$$(A - B)^2 = A^2 - 2AB + B^2$$

$$2AB = isx$$

$$\text{Here } A = ax, B = \frac{is}{2a}$$



$$\begin{aligned}
 &= \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} du \quad \because i^2 = -1 \\
 &= \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} 2 \int_0^{\infty} e^{-u^2} du \quad \because e^{-u^2} \text{ is an even function} \\
 &= \frac{1}{a\sqrt{2}\sqrt{\pi}} e^{\frac{-s^2}{4a^2}} 2 \frac{\sqrt{\pi}}{2} \quad \because \int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}
 \end{aligned}$$

$$\boxed{F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}}} \quad \text{-----(1)}$$

Deduction:

To prove $e^{-\frac{x^2}{2}}$ is self-reciprocal

It is enough to prove that $F\left[e^{-\frac{x^2}{2}}\right]$ is $e^{-\frac{s^2}{2}}$

Put $a = \frac{1}{\sqrt{2}}$ in (1)

$$F\left[e^{-\left(\frac{1}{\sqrt{2}}\right)^2 x^2}\right] = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\sqrt{2}} e^{-\frac{s^2}{4\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$F\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{s^2}{2}}$$

$$\boxed{F\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{s^2}{2}}}$$

$\therefore e^{-\frac{x^2}{2}}$ is self-reciprocal.

Find the Fourier cosine transform of $e^{-a^2x^2}$ Hence find $F_s[xe^{-a^2x^2}]$.

Solution:

Let $f(x) = e^{-a^2x^2}$

The Fourier cosine transform f(x) is

$$\begin{aligned}
 F_c[f(x)] &= F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx & \because \int_0^{\infty} f(x) dx &= \frac{1}{2} \int_{-\infty}^{\infty} f(x) dx \\
 &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-a^2x^2} \cos sx \, dx
 \end{aligned}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^{\infty} e^{-a^2x^2} \cos sx \, dx$$

$$F_c[f(x)] = \text{R.P. of } \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} \, dx \quad \because \cos sx = \text{R.P. of } e^{isx}$$

$$F_c[f(x)] = \text{R.P. of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} \, dx$$



$$\begin{aligned}
 &= \text{R.P. of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx \\
 &= \text{R.P. of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2+isx} dx \\
 &= \text{R.P. of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2-isx)} dx \\
 &= \text{R.P. of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[(ax)^2 - isx + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2 \right]} dx \\
 &= \text{R.P. of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{\left(\frac{is}{2a}\right)^2} dx \\
 &= \text{R.P. of } \frac{1}{\sqrt{2\pi}} e^{\left(\frac{is}{2a}\right)^2} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx
 \end{aligned}$$

$$\begin{aligned}
 (a-b)^2 &= a^2 - 2ab + b^2 \\
 -2ab &= isx \\
 \text{Here } a &= ax \\
 -2axb &= -isx \Rightarrow b = \frac{is}{2a}
 \end{aligned}$$

Let $u = ax - \frac{is}{2a} \Rightarrow du = adx \Rightarrow dx = \frac{du}{a}$; $u : -\infty \text{ to } \infty$

$$\begin{aligned}
 &= \text{R.P. of } \frac{1}{\sqrt{2\pi}} e^{\frac{i^2s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a} \\
 &= \text{R.P. of } \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} du \quad \because i^2 = -1 \\
 &= \text{R.P. of } \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} 2 \int_0^{\infty} e^{-u^2} du \quad \because e^{-u^2} \text{ is an even function} \\
 &= \text{R.P. of } \frac{1}{a\sqrt{2}\sqrt{\pi}} e^{\frac{-s^2}{4a^2}} 2 \frac{\sqrt{\pi}}{2} \quad \because \int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}
 \end{aligned}$$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}} \quad \text{-----(1)}$$

Deduction:

$$F_s [xf(x)] = -\frac{d}{ds} \{F_c[f(x)]\} = -\frac{d}{ds} [F_c(s)]$$

$$\begin{aligned}
 F_s [xe^{-a^2x^2}] &= -\frac{d}{ds} \left\{ F_c \left[e^{-a^2x^2} \right] \right\} \\
 &= -\frac{d}{ds} \left[\frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}} \right] \\
 &= -\frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}} \left(\frac{-2s}{4a^2} \right)
 \end{aligned}$$

$$F_s [xe^{-a^2x^2}] = \frac{s}{2\sqrt{2}a^3} e^{\frac{-s^2}{4a^2}}$$

Solve for $f(x)$, the integral equation $\int_0^{\infty} f(x) \sin sxdx = \begin{cases} 1, 0 \leq s < 1 \\ 2, 1 \leq s < 2. \\ 0, s \geq 2 \end{cases}$

Solution:



$$\text{Given } \int_0^{\infty} f(x) \sin sx dx = \begin{cases} 1, & 0 \leq s < 1 \\ 2, & 1 \leq s < 2 \\ 0, & s \geq 2 \end{cases} \text{-----(1)}$$

We know that

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} F_s^{-1} \begin{cases} 1, & 0 \leq s < 1 \\ 2, & 1 \leq s < 2 \\ 0, & s \geq 2 \end{cases}$$

$$F^{-1}[F_s(s)] = f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(s) \sin sx ds$$

$$f(x) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \left[\int_0^1 1 \sin sx ds + \int_1^2 2 \sin sx ds + \int_2^{\infty} 0 \sin sx ds \right]$$

$$= \frac{2}{\pi} \left[\int_0^1 1 \sin sx ds + \int_1^2 2 \sin sx ds \right]$$

$$= \frac{2}{\pi} \left[\left(\frac{-\cos sx}{x} \right)_0^1 + 2 \left(\frac{-\cos sx}{x} \right)_1^2 \right]$$

$$= \frac{2}{\pi} \left[\left(\frac{-\cos x}{x} + \frac{\cos 0}{x} \right) + 2 \left(\frac{-\cos 2x}{x} + \frac{\cos x}{x} \right) \right]$$

$$= \frac{2}{\pi} \left[\frac{-\cos x}{x} + \frac{1}{x} - 2 \frac{\cos 2x}{x} + 2 \frac{\cos x}{x} \right]$$

$$= \frac{2}{\pi x} [1 - \cos x - 2 \cos 2x + 2 \cos x]$$

$$f(x) = \frac{2}{\pi x} [1 + \cos x - 2 \cos 2x]$$

Find the Fourier cosine and sine transform of x^{n-1} . Hence show that $\frac{1}{\sqrt{x}}$ is self-reciprocal under

Fourier cosine and sine transforms.

Solution:

By definition of Gamma integral

$$\int_0^{\infty} e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}, \quad a > 0, n > 0$$

Put $a = is$

$$\int_0^{\infty} e^{-isx} x^{n-1} dx = \frac{\Gamma n}{(is)^n}, \quad a > 0, n > 0$$

$$\int_0^{\infty} x^{n-1} e^{-isx} dx = \frac{\Gamma n}{i^n s^n}$$

$$= \frac{\Gamma n}{s^n} (-i)^n$$

$$= \frac{\Gamma n}{s^n} \left(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \right)^n \quad \because e^{-i\frac{\pi}{2}} = \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} = -i$$

$$= \frac{\Gamma n}{s^n} \left(\cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2} \right) \quad \because \text{by Demorive's theorem } (\cos \theta \pm i \sin \theta)^n = \cos n\theta \pm i \sin n\theta$$

$$\int_0^{\infty} x^{n-1} (\cos sx - i \sin sx) dx = \frac{\Gamma n}{s^n} \left(\cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2} \right)$$



$$\int_0^{\infty} x^{n-1} \cos sx \, dx - i \int_0^{\infty} x^{n-1} \sin sx \, dx = \frac{\Gamma n}{s^n} \cos \frac{n\pi}{2} - i \frac{\Gamma n}{s^n} \sin \frac{n\pi}{2}$$

Equating real and imaginary parts on both sides

$$\int_0^{\infty} x^{n-1} \cos sx \, dx = \frac{\Gamma n}{s^n} \cos \frac{n\pi}{2} \qquad \int_0^{\infty} x^{n-1} \sin sx \, dx = \frac{\Gamma n}{s^n} \sin \frac{n\pi}{2}$$

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{n-1} \cos sx \, dx = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \cos \frac{n\pi}{2} \qquad \sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{n-1} \sin sx \, dx = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \sin \frac{n\pi}{2}$$

$$F_c [x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \cos \frac{n\pi}{2}$$

$$F_s [x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \sin \frac{n\pi}{2}$$

Deduction:

To prove $\frac{1}{\sqrt{x}}$ is self-reciprocal under Fourier cosine and sine transforms.

It is enough to prove that $F_c \left[\frac{1}{\sqrt{x}} \right] = \frac{1}{\sqrt{s}}$ and $F_s \left[\frac{1}{\sqrt{x}} \right] = \frac{1}{\sqrt{s}}$

We know that

$$F_c [x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \cos \frac{n\pi}{2} \qquad F_s [x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \sin \frac{n\pi}{2}$$

Put $n = \frac{1}{2}$

$$F_c \left[x^{\frac{1}{2}-1} \right] = \sqrt{\frac{2}{\pi}} \frac{\Gamma \left(\frac{1}{2} \right)}{s^{\frac{1}{2}}} \cos \frac{\pi}{4}$$

$$F_s \left[x^{\left(\frac{1}{2}-1 \right)} \right] = \sqrt{\frac{2}{\pi}} \frac{\Gamma \left(\frac{1}{2} \right)}{s^{\frac{1}{2}}} \sin \frac{\pi}{4}$$

$$F_c \left[x^{-\frac{1}{2}} \right] = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{s}} \frac{1}{\sqrt{2}}$$

$$F_s \left[x^{-\frac{1}{2}} \right] = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{s}} \frac{1}{\sqrt{2}} \quad \because \cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \Gamma \left(\frac{1}{2} \right) = \sqrt{\pi}$$

$$F_c \left[\frac{1}{\sqrt{x}} \right] = \frac{1}{\sqrt{s}}$$

$$F_s \left[\frac{1}{\sqrt{x}} \right] = \frac{1}{\sqrt{s}}$$

$\therefore \frac{1}{\sqrt{x}}$ is self-reciprocal under Fourier cosine and sine transforms.

Find the function $f(x)$ if its sine transform is $\frac{e^{-as}}{s}$

Solution:

Given $F_s [f(x)] = F_s(s) = \frac{e^{-as}}{s}$

$$f(x) = F^{-1} [F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(s) \sin sx \, ds$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-as}}{s} \sin sx \, ds$$

Taking diff on both sides w.r.to x

$$\begin{aligned} \frac{d}{dx} [f(x)] &= \frac{d}{dx} \left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-as}}{s} \sin sx \, ds \right] \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-as}}{s} \frac{\partial}{\partial x} (\sin sx) \, ds \end{aligned}$$



$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-as}}{s} \cos sx \times s \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-as} \cos sx \, ds$$

$$\frac{d}{dx}[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + x^2} \right] \quad \because \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + s^2} \text{ here } a = a, b = x$$

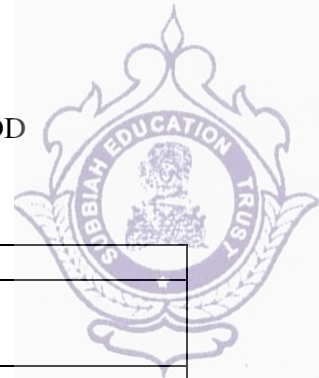
Integrating on w.r.to x

$$f(x) = \sqrt{\frac{2}{\pi}} a \int \frac{1}{a^2 + x^2} \quad \because \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$$

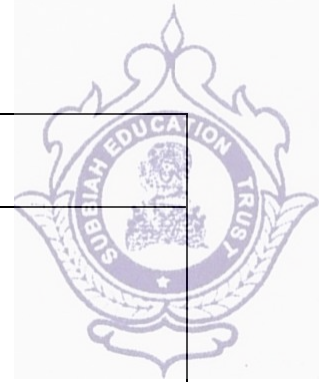
$$= \sqrt{\frac{2}{\pi}} a \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$$

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1} \left(\frac{x}{a} \right)}$$

5.1 INVERSE Z-TRANSFORMS BY PARTIAL FRACTION METHOD



Z-Transform of some basic functions:	
1.	$Z[a^n] = \frac{z}{z-a} \quad ; \quad Z[1] = \frac{z}{z-1} \quad ; \quad Z[(-a)^n] = \frac{z}{z+a}$
2.	$Z[n] = \frac{z}{(z-1)^2}$
3.	$Z\left[\frac{1}{n}\right] = \log\left(\frac{z}{z-1}\right)$
4.	$Z\left[\frac{1}{n+1}\right] = z \log\left(\frac{z}{z-1}\right)$
5.	$Z\left[\frac{1}{n-1}\right] = \frac{1}{z} \log\left(\frac{z}{z-1}\right)$
6.	$Z\left[\frac{1}{n!}\right] = e^{\frac{1}{z}}$
7.	$Z[\cos n\theta] = \frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1}$
8.	$Z[\sin n\theta] = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$
Inverse Z-Transforms:	
The inverse Z-transform of $Z[f(n)] = F(z)$ is defined as $f(n) = Z^{-1}[F(z)]$.	
The inverse Z-Transform of some basic functions:	
1.	$Z^{-1}\left[\frac{z}{z-1}\right] = 1 \quad ; \quad Z^{-1}\left[\frac{z}{z+1}\right] = (-1)^n$
2.	$Z^{-1}\left[\frac{z}{z-a}\right] = a^n \quad ; \quad Z^{-1}\left[\frac{z}{z+a}\right] = (-a)^n \quad ; \quad Z^{-1}\left[\frac{1}{z+a}\right] = a^{n-1}$
3.	$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = (n+1)a^n$ For Eg. 1) $Z^{-1}\left[\frac{z}{(z-a)^2}\right] = (n-1+1)a^{n-1} = na^{n-1}$ 2) $Z^{-1}\left[\frac{1}{(z-a)^2}\right] = (n-2+1)a^{n-2} = (n-1)a^{n-2}$ 3) $Z^{-1}\left[\frac{z^2}{(z-1)^2}\right] = (n+1)1^n = n+1$ 4) $Z^{-1}\left[\frac{z}{(z-1)^2}\right] = (n-1+1)1^n = n$ 5) $Z^{-1}\left[\frac{1}{(z-1)^2}\right] = (n-2+1)1^n = n-1$
4.	$Z^{-1}\left[\frac{z^2}{z^2+a^2}\right] = a^n \cos \frac{n\pi}{2}$



$$5. \quad Z^{-1} \left[\frac{z}{z^2 + a^2} \right] = a^n \cos(n-1) \frac{\pi}{2} = a^n \cos \left(\frac{\pi}{2} - \frac{n\pi}{2} \right) = a^n \sin \frac{n\pi}{2}$$

Finding Inverse Z-transform by method of **Partial Fractions:**

Rules of Partial Fractions:

1. Denominator containing Linear factors:

$$\frac{f(z)}{(z-a)(z-b)(z-c)\dots} = \frac{A}{(z-a)} + \frac{B}{(z-b)} + \frac{C}{(z-c)} + \dots$$

2. Denominator containing factors $(z-a)^n$:

$$\frac{f(z)}{(z-a)^n} = \frac{A}{(z-a)} + \frac{B}{(z-a)^2} + \frac{C}{(z-a)^3} + \dots + \frac{D}{(z-a)^n}$$

3. Denominator contains a quadratic factor of the form $az^2 + bz + c$ (where a,b,c are constants):

$$\frac{f(z)}{az^2 + bz + c} = \frac{A}{az^2 + bz + c} + \frac{Bz}{az^2 + bz + c}$$

(Or)
$$\frac{f(z)}{az^2 + bz + c} = \frac{Az + B}{az^2 + bz + c}$$

1. Find $Z^{-1} \left[\frac{z}{(z+1)(z-1)^2} \right]$ using the method partial fraction.

Solution:

$$F(z) = \frac{z}{(z+1)(z-1)^2}$$

$$\frac{F(z)}{z} = \frac{1}{(z+1)(z-1)^2} \text{ ----- (1)}$$

Now,

$$\frac{1}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$1 = A(z-1)^2 + B(z+1)(z-1) + C(z+1)$$

Put $z = 1 \Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$

Put $z = -1, \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$

Put $z = 0 \Rightarrow 1 = A - B + C \Rightarrow B = \frac{1}{4} + \frac{1}{2} - 1 \Rightarrow B = \frac{1+2-4}{4} \Rightarrow B = \frac{-1}{4}$

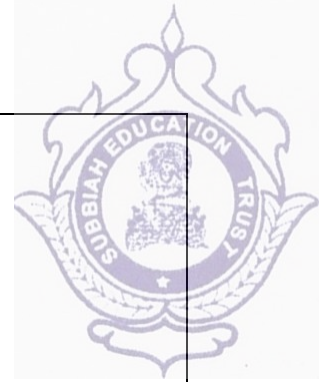
$$\frac{1}{(z+1)(z-1)^2} = \frac{\frac{1}{4}}{z+1} + \frac{\frac{-1}{4}}{z-1} + \frac{\frac{1}{2}}{(z-1)^2}$$

$$(1) \Rightarrow F(z) = \frac{1}{4} \frac{z}{z+1} - \frac{1}{4} \frac{z}{z-1} + \frac{1}{2} \frac{z}{(z-1)^2}$$

Taking Z^{-1} on both sides

$$(1) \Rightarrow Z^{-1}[F(z)] = \frac{1}{4} Z^{-1} \left[\frac{z}{z+1} \right] - \frac{1}{4} Z^{-1} \left[\frac{z}{z-1} \right] + \frac{1}{2} Z^{-1} \left[\frac{z}{(z-1)^2} \right]$$

$$f(n) = \frac{1}{4}(-1)^n - \frac{1}{4}(1) + \frac{1}{2}n$$



2. Find $Z^{-1} \left[\frac{z^{-2}}{(1+z^{-1})^2(1-z^{-1})} \right]$.

Solution:

$$F(z) = \frac{z^{-2}}{(1+z^{-1})^2(1-z^{-1})} = \frac{1}{\cancel{z} \frac{(z+1)^2}{\cancel{z}} \left(\frac{z-1}{z} \right)}$$

$$F(z) = \frac{z}{(z+1)^2(z-1)}$$

$$\frac{F(z)}{z} = \frac{1}{(z+1)^2(z-1)} \text{ ----- (1)}$$

$$\frac{1}{(z-1)(z+1)^2} = \frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{(z+1)^2}$$

$$1 = A(z+1)^2 + B(z-1)(z+1) + C(z-1)$$

Put $z = 1, 1 = 4A \Rightarrow \boxed{A = \frac{1}{4}}$

Put $z = -1, 1 = -2c \Rightarrow \boxed{c = -\frac{1}{2}}$

Equating co-efficients of $z^2 \Rightarrow 0 = A + B \Rightarrow \boxed{B = -\frac{1}{4}}$

$$(1) \Rightarrow \frac{F(z)}{z} = \frac{1}{4} \frac{1}{z-1} + \frac{-1}{4} \frac{1}{z+1} - \frac{1}{2} \frac{1}{(z+1)^2}$$

$$(1) \Rightarrow Z^{-1}[F(z)] = \frac{1}{4} Z^{-1} \left[\frac{z}{z-1} \right] - \frac{1}{4} Z^{-1} \left[\frac{z}{z+1} \right] - \frac{1}{2} Z^{-1} \left[\frac{z}{(z+1)^2} \right]$$

$$f(n) = \frac{1}{4}(1)^n - \frac{1}{4}(-1)^n + \frac{1}{2}n(-1)^n$$

$$\boxed{f(n) = \frac{1}{4} - \frac{1}{4}(-1)^n + \frac{1}{2}n(-1)^n}$$

3. Find $Z^{-1} \left[\frac{z^{-2}}{(1-z^{-1})(1-2z^{-1})(1-3z^{-1})} \right]$.

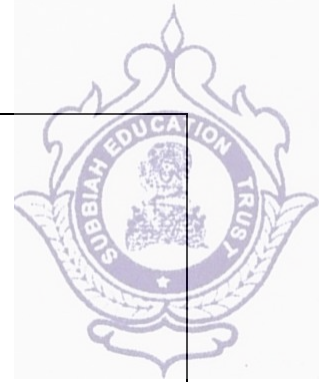
Solution:

$$F(z) = \left[\frac{z^{-2}}{(1-z^{-1})(1-2z^{-1})(1-3z^{-1})} \right] = \frac{\frac{1}{z^2}}{\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)\left(1-\frac{3}{z}\right)}$$

$$= \frac{1}{\cancel{z} \left(\frac{z-1}{\cancel{z}} \right) \left(\frac{z-2}{\cancel{z}} \right) \left(\frac{z-3}{z} \right)}$$

$$F(z) = \frac{z}{(z-1)(z-2)(z-3)}$$

$$\frac{F(z)}{z} = \frac{1}{(z-1)(z-2)(z-3)} \text{ ----- (1)}$$



Now by Partial Fraction,

$$\frac{1}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$1 = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2)$$

Put $z = 2, \Rightarrow 1 = -B \Rightarrow \boxed{B = -1}$

Put $z = 1, \Rightarrow 1 = 2A \Rightarrow \boxed{A = \frac{1}{2}}$

Put $z = 3, \Rightarrow 1 = 2C \Rightarrow \boxed{C = \frac{1}{2}}$

$$(1) \Rightarrow F(z) = \frac{1}{2} \frac{z}{z-1} - \frac{z}{z-2} + \frac{1}{2} \frac{z}{z-3}$$

$$(1) \Rightarrow Z^{-1}[F(z)] = \frac{1}{2} Z^{-1} \left[\frac{z}{z-1} \right] - Z^{-1} \left[\frac{z}{z-2} \right] + \frac{1}{2} Z^{-1} \left[\frac{z}{z-3} \right]$$

$$f(n) = \frac{1}{2}(1)^n - (2)^n + \frac{1}{2}(3)^n$$

$$\boxed{f(n) = \frac{1}{2} - 2^n + \frac{1}{2} 3^n}$$

4. Find the Z-transform of $\frac{z^2 + z}{(z-1)(z^2 + 1)}$ using partial fraction.

Solution:

$$F(z) = \frac{z^2 + z}{(z-1)(z^2 + 1)}$$

$$\frac{F(z)}{z} = \frac{z+1}{(z-1)(z^2 + 1)}$$

$$\frac{z+1}{(z-1)(z^2 + 1)} = \frac{A}{z-1} + \frac{B}{z^2 + 1} + \frac{Cz}{z^2 + 1}$$

$$z+1 = A(z^2 + 1) + B(z-1) + Cz(z-1)$$

Put $z = 1, \Rightarrow 2 = 2A \Rightarrow \boxed{A = 1}$

Equating co-efficients of $z^2 \Rightarrow 0 = A + C \Rightarrow \boxed{C = -1}$

Put $z = 0, \Rightarrow 1 = A - B \Rightarrow B = A - 1 = 1 - 1 = 0 \Rightarrow \boxed{B = 0}$

$$\frac{F(z)}{z} = \frac{1}{z-1} + \frac{0}{z^2 + 1} + \frac{-z}{z^2 + 1}$$

$$F(z) = \frac{z}{z-1} - \frac{z^2}{z^2 + 1}$$

Put Z^{-1} on both sides

$$Z^{-1}[F(z)] = Z^{-1} \left[\frac{z}{z-1} \right] - Z^{-1} \left[\frac{z^2}{z^2 + 1} \right]$$

$$\boxed{f(n) = 1 - \cos \frac{n\pi}{2}} \quad \because Z^{-1} \left[\frac{z^2}{z^2 + a^2} \right] = \cos \frac{n\pi}{2}$$



5.2 Finding Inverse Z-transform by Residue Method

Procedure:

1. write $F(z)$ from given expression and write $F(z)z^{n-1}$
2. Find the poles by equating denominator to zero in $F(z)z^{n-1}$
3. Write the order of poles
4. Find the residue at these poles

Case i: If $z = a$ is pole of order 1 (or) simple pole then

$$\left[\text{Res } F(z)z^{n-1} \right]_{z=a} = \lim_{z \rightarrow a} (z-a)F(z)z^{n-1}$$

1 Find $Z^{-1} \left[\frac{2z}{(z-2)(z^2+1)} \right]$ by the method of residues.

Solution:

$$\text{Let } F(z) = \frac{2z}{(z-1)(z^2+1)}$$

$$F(z)z^{n-1} = \frac{2z^n}{(z-1)(z^2+1)}$$

$$F(z)z^{n-1} = \frac{2z^n}{(z-1)(z+i)(z-i)} \text{ ----- (1)}$$

Here $z = 1, z = i$ and $z = -i$ are poles of order 1.

$$1) \text{ Res } \left[F(z)z^{n-1} \right]_{z=1} = \lim_{z \rightarrow 1} (z-1)F(z)z^{n-1}$$

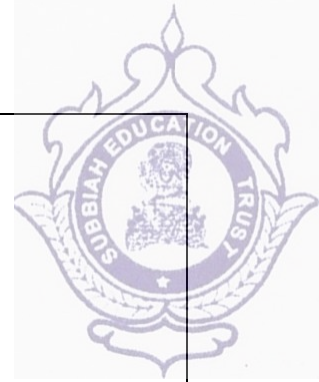
$$\begin{aligned} \text{Res } \left[F(z)z^{n-1} \right]_{z=1} &= \lim_{z \rightarrow 1} \cancel{(z-1)} \frac{2z^n}{\cancel{(z-1)}(z+i)(z-i)} \\ &= \lim_{z \rightarrow 1} \frac{2z^n}{(z+i)(z-i)} \\ &= \frac{2(1)^n}{(1+i)(1-i)} \\ &= \frac{2}{2} \quad \because (1+i)(1-i) = 1^2 - i^2 = 1 - (-1) = 1 + 1 = 2 \end{aligned}$$

$$\boxed{\text{Res } \left[F(z)z^{n-1} \right]_{z=1} = 1}$$

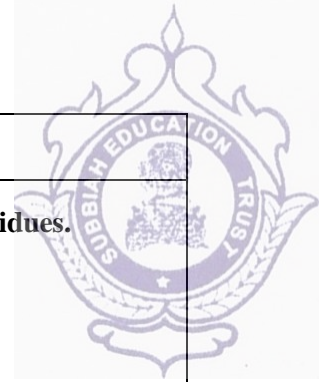
$$2) \text{ Res } \left[F(z)z^{n-1} \right]_{z=i} = \lim_{z \rightarrow i} (z-i)F(z)z^{n-1}$$

$$\begin{aligned} \text{Res } \left[F(z)z^{n-1} \right]_{z=i} &= \lim_{z \rightarrow i} \cancel{(z-i)} \frac{2z^n}{(z-1)\cancel{(z-i)}(z+i)} \\ &= \lim_{z \rightarrow i} \frac{2z^n}{(z-1)(z+i)} \\ &= \frac{2(i)^n}{(i-1)(i+i)} \\ &= \frac{2(i)^n}{2i(i-1)} \\ &= \frac{(i)^n}{i(i-1)} = \frac{(i)^n}{(i^2-i)} = \frac{(i)^n}{(-1-i)} \end{aligned}$$

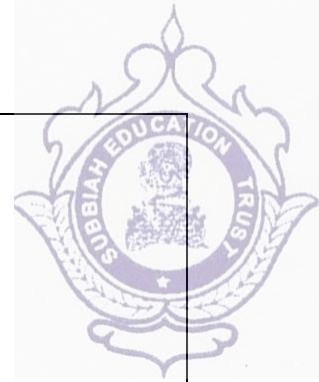
$$\boxed{\text{Res } \left[F(z)z^{n-1} \right]_{z=i} = \frac{-(i)^n}{(1+i)}}$$



	<p>3) $\text{Res} \left[F(z)z^{n-1} \right]_{z=-i} = \lim_{z \rightarrow -i} (z+i)F(z)z^{n-1}$</p> $\text{Res} \left[F(z)z^{n-1} \right]_{z=-i} = \lim_{z \rightarrow -i} \frac{2z^n}{(z-1)(z+i)(z-i)}$ $= \lim_{z \rightarrow -i} \frac{2z^n}{(z-1)(z-i)}$ $= \frac{2(-i)^n}{(-i-1)(-i-i)} = \frac{2(-i)^n}{(1+i)(2i)}$ $= \frac{(-i)^n}{(1+i)(i)} = \frac{(-i)^n}{(i+i^2)} = \frac{(-i)^n}{(i-1)}$ <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $\text{Res} \left[F(z)z^{n-1} \right]_{z=-i} = \frac{(-i)^n}{(i-1)}$ </div> <p>$f(n) = \text{sum of residues of } F(z)z^{n-1}$</p> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $f(n) = 1 - \frac{(i)^n}{(1+i)} + \frac{(-i)^n}{(i-1)}$ </div>
<p>2.</p>	<p>Find the inverse Z-Transform of $\frac{z(z+1)}{(z-1)^3}$ by residue method.</p> <p>Solution:</p> <p>Let $F(z) = \frac{z(z+1)}{(z-1)^3}$</p> $F(z)z^{n-1} = \frac{zz^{n-1}(z+1)}{(z-1)^3}$ $F(z)z^{n-1} = \frac{z^n(z+1)}{(z-1)^3} \text{ ----- (1)}$ <p>$z = 1$ is a pole of order 3</p> $\text{Res} \left[F(z)z^{n-1} \right]_{z=a} = \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m F(z)z^{n-1}$ $\text{Res} \left[F(z)z^{n-1} \right]_{z=1} = \frac{1}{(3-1)!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \frac{z^n(z+1)}{(z-1)^3}$ $= \frac{1}{2} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} [z^{n+1} + z^n]$ $= \frac{1}{2} \lim_{z \rightarrow 1} \frac{d}{dz} [(n+1)z^n + nz^{n-1}]$ $= \frac{1}{2} \lim_{z \rightarrow 1} [(n+1)nz^{n-1} + n(n-1)z^{n-2}]$ $= \frac{1}{2} \lim_{z \rightarrow 1} [(n^2+n)(1)^{n-1} + (n^2-n)1^{n-2}]$ $= \frac{1}{2} [n^2 + n + n^2 - n]$ $\text{Res} \left[F(z)z^{n-1} \right]_{z=1} = \frac{1}{2} [2n^2]$ $\text{Res} \left[F(z)z^{n-1} \right]_{z=1} = n^2$



	$f(n) = \text{sum of residues of } F(z)z^{n-1} = n^2$
<p>3.</p>	<p>Find the inverse Z-transform of the function $\frac{z}{z^2 + 7z + 10}$ by the method of residues.</p> <p>Solution:</p> $Z^{-1} \left[\frac{z}{z^2 + 7z + 10} \right] = ?$ $F(z) = \frac{z}{z^2 + 7z + 10} = \frac{z}{(z+2)(z+5)}$ $F(z)z^{n-1} = \frac{zz^{n-1}}{(z+2)(z+5)}$ $F(z)z^{n-1} = \frac{z^n}{(z+2)(z+5)} \text{ -----(1)}$ <p>Here $z=-2$ and $z=-5$ are pole of order 1</p> <p>1) $\text{Res} \left[F(z)z^{n-1} \right]_{z=-2} = \lim_{z \rightarrow -2} (z+2)F(z)z^{n-1}$</p> $\text{Res} \left[F(z)z^{n-1} \right]_{z=-2} = \lim_{z \rightarrow -2} (z+2) \frac{z^n}{(z+2)(z+5)}$ $= \frac{(-2)^n}{(-2+5)} = \frac{(-2)^n}{3}$ <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $\text{Res} \left[F(z)z^{n-1} \right]_{z=-2} = \frac{(-2)^n}{3}$ </div> <p>2) $\text{Res} \left[F(z)z^{n-1} \right]_{z=-5} = \lim_{z \rightarrow -5} (z+5) \frac{z^n}{(z+2)(z+5)}$</p> $= \frac{(-5)^n}{(-5+2)} = \frac{(-5)^n}{-3}$ <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $\text{Res} \left[F(z)z^{n-1} \right]_{z=-5} = \frac{-(-5)^n}{3}$ </div> <p>$f(n) = \text{sum of residues of } F(z)z^{n-1}$</p> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $f(n) = \frac{(-2)^n}{3} - \frac{(-5)^n}{3} = \frac{1}{3} [(-2)^n - (-5)^n]$ </div>
<p>4.</p>	<p>Find $Z^{-1} \left[\frac{z^{-2}}{(1+z^{-1})^2(1-z^{-1})} \right]$ by using residue method.</p> <p>Solution:</p> $F(z) = \frac{z^{-2}}{(1+z^{-1})^2(1-z^{-1})} = \frac{1}{z^2 \left(\frac{z+1}{z} \right)^2 \left(\frac{z-1}{z} \right)}$ $F(z) = \frac{z}{(z+1)^2(z-1)}$ $F(z)z^{n-1} = \frac{zz^{n-1}}{(z+1)^2(z-1)}$



$$F(z)z^{n-1} = \frac{z^n}{(z+1)^2(z-1)} \text{ ----- (1)}$$

Here $z = -1$ is pole of order 2, and $z = 1$ is pole of order 1

$$1) \operatorname{Res} [F(z)z^{n-1}]_{z=-1} = \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m F(z)z^{n-1}$$

$$\begin{aligned} \operatorname{Res} [F(z)z^{n-1}]_{z=-1} &= \frac{1}{(2-1)!} \lim_{z \rightarrow -1} \frac{d^2}{dz^2} \frac{z^n}{(z+1)^2(z-1)} \\ &= \frac{1}{1!} \lim_{z \rightarrow -1} \frac{d}{dz} \left[\frac{z^n}{z-1} \right] \\ &= \lim_{z \rightarrow -1} \left[\frac{(z-1)nz^{n-1} - z^n(1-0)}{(z-1)^2} \right] \\ &= \frac{(-1-1)n(-1)^{n-1} - (-1)^n}{(-1-1)^2} = \frac{-2n(-1)^{n-1} - (-1)^n}{4} = \frac{(-1)^n}{4} [2n-1] \end{aligned}$$

$$\operatorname{Res} [F(z)z^{n-1}]_{z=-1} = \frac{(-1)^n}{4} [2n-1]$$

$$2) \operatorname{Res} [F(z)z^{n-1}]_{z=1} = \lim_{z \rightarrow a} (z-a)F(z)z^{n-1}$$

$$\begin{aligned} \operatorname{Res} [F(z)z^{n-1}]_{z=1} &= \lim_{z \rightarrow 1} \frac{z^n}{(z+1)^2} \\ &= \lim_{z \rightarrow 1} \frac{z^n}{(z+1)^2} = \frac{1^n}{(1+1)^2} = \frac{1}{2} \end{aligned}$$

$$\operatorname{Res} [F(z)z^{n-1}]_{z=1} = \frac{1}{2}$$

$f(n) = \text{sum of residues of } F(z)z^{n-1}$

$$f(n) = \frac{(-1)^n}{4} [2n-1] + \frac{1}{2}$$

5. Using complex residue theorem evaluate $Z^{-1} \left[\frac{9z^3}{(3z-1)^2(z-2)} \right]$.

Solution:

$$Z^{-1} \left[\frac{9z^3}{(3z-1)^2(z-2)} \right] = Z^{-1} \left[\frac{9z^3}{9(z-\frac{1}{3})^2(z-2)} \right] = Z^{-1} \left[\frac{z^3}{(z-\frac{1}{3})^2(z-2)} \right]$$

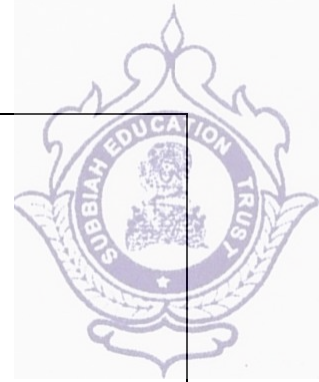
$$F(z) = \frac{z^3}{(z-\frac{1}{3})(z-2)}$$

$$F(z)z^{n-1} = \frac{z^3 z^{n-1}}{(z-\frac{1}{3})^2(z-2)}$$

$$F(z)z^{n-1} = \frac{z^{n+2}}{(z-\frac{1}{3})^2(z-2)}$$

Here $z = \frac{1}{3}$ are pole of order 2 and $z = 2$ is simple pole.

$$1) \operatorname{Res} [F(z)z^{n-1}]_{z=a} = \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m F(z)z^{n-1} \quad \text{here } m=2$$



$$\begin{aligned}
 \text{Res}\left[F(z)z^{n-1}\right]_{z=\frac{1}{3}} &= \lim_{z \rightarrow \frac{1}{3}} \frac{d}{dz} \left[\frac{z^{n+2}}{(z-\frac{1}{3})^2(z-2)} \right] \\
 &= \lim_{z \rightarrow \frac{1}{3}} \frac{d}{dz} \left[\frac{z^{n+2}}{z-2} \right] \\
 &= \lim_{z \rightarrow \frac{1}{3}} \left[\frac{(z-2)(n+2)z^{n+1} - z^{n+2}(1)}{(z-2)^2} \right] \\
 &= \lim_{z \rightarrow \frac{1}{3}} \left\{ \frac{z^{n+1}[(z-2)(n+2) - z]}{(z-2)^2} \right\} \\
 &= \frac{\left(\frac{1}{3}\right)^{n+1} \left[\left(\frac{1}{3} - 2\right)(n+2) - \frac{1}{3} \right]}{\left(\frac{1}{3} - 2\right)^2} \\
 \text{Res}\left[F(z)z^{n-1}\right]_{z=\frac{1}{3}} &= \frac{\left(\frac{1}{3}\right)^{n+1} \left[\left(\frac{-5(n+2)}{3} - \frac{1}{3}\right) \right]}{\left(\frac{-5}{3}\right)^2} = \frac{\left(\frac{1}{3}\right)^{n+1} \left(\frac{-5n-10-1}{3}\right)}{\frac{25}{9}} \\
 &= \frac{9}{25} \left(\frac{1}{3}\right)^n \left(\frac{1}{3}\right) \left(\frac{-5n-11}{3}\right) = \frac{-1}{25} \left(\frac{1}{3}\right)^n (5n+11)
 \end{aligned}$$

$$\boxed{\text{Res}\left[F(z)z^{n-1}\right]_{z=\frac{1}{3}} = \frac{-1}{25} \left(\frac{1}{3}\right)^n (5n+11)}$$

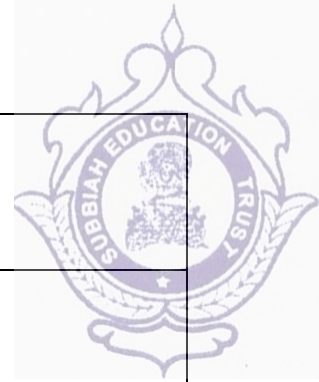
$$2) \text{Res}\left[F(z)z^{n-1}\right]_{z=2} = \lim_{z \rightarrow 2} \left(\frac{z^{n+2}}{(z-\frac{1}{3})^2(z-2)} \right)$$

$$\text{Res}\left[F(z)z^{n-1}\right]_{z=2} = \frac{2^{n+2}}{\left(2-\frac{1}{3}\right)^2} = \frac{9}{25} 2^{n+2}$$

$$\boxed{\text{Res}\left[F(z)z^{n-1}\right]_{z=2} = \frac{9}{25} 2^{n+2}}$$

$f(n) = \text{sum of residues of } F(z)z^{n-1}$

$$\boxed{f(n) = f(n) = \frac{9}{25} 2^{n+2} + \frac{-1}{25} \left(\frac{1}{3}\right)^n (5n+11)}$$



5.3 Finding Inverse Z-transform by Convolution theorem

1. Find inverse Z-transform of $\frac{z^2}{(z-a)^2}$ by using convolution theorem.

Solution:

Given $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = ?$ $f(n) * g(n) = \sum_{k=0}^n f(k)g(n-k)$

By convolution theorem

$$Z^{-1}[F(z) \cdot G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$$

$$\begin{aligned} Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] &= Z^{-1}\left[\frac{z}{z-a} \cdot \frac{z}{z-a}\right] \\ &= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right] \\ &= a^n * a^n \\ &= \sum_{k=0}^n a^k a^{n-k} \quad \because f(n) * g(n) = \sum_{k=0}^n f(k)g(n-k) \\ &= \sum_{k=0}^n \cancel{a^k} \cancel{a^{n-k}} a^n \\ &= a^n \sum_{k=0}^n 1 \end{aligned}$$

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = a^n(n+1) \cdot 1 = (n+1)a^n$$

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = (n+1)a^n$$

2. By using convolution theorem, show that the inverse Z-transform of $\frac{z^2}{(z+a)(z+b)}$ is

$$\frac{(-1)^n}{b-a} [b^{n+1} - a^{n+1}]$$

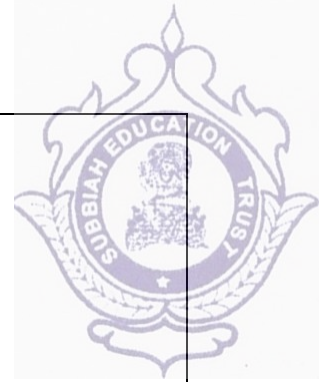
Solution:

Given $Z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right] = ?$

By convolution theorem

$$Z^{-1}[F(z) \cdot G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$$

$$\begin{aligned} Z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right] &= Z^{-1}\left[\frac{z}{z+a} \cdot \frac{z}{z+b}\right] \\ &= Z^{-1}\left[\frac{z}{z+a}\right] * Z^{-1}\left[\frac{z}{z+b}\right] \\ &= (-a)^n * (-b)^n \\ &= \sum_{k=0}^n (-a)^k (-b)^{n-k} \quad \because f(n) * g(n) = \sum_{k=0}^n f(k)g(n-k) \end{aligned}$$



$$\begin{aligned}
 &= (-1)^n \sum_{k=0}^n a^k b^{-k} b^n \\
 &= (-1)^n b^n \sum_{k=0}^n \left(\frac{a}{b}\right)^k \\
 &= (-1)^n b^n \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^3 + \dots + \left(\frac{a}{b}\right)^n \right] \\
 &= (-1)^n b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1} \right] = b^n \left[\frac{a^{n+1} - b^{n+1}}{b^{n+1} - 1} \right] = b^n \left[\frac{a^{n+1} - b^{n+1}}{b^{n+1} \left(\frac{a-b}{b}\right)} \right] \\
 &= (-1)^n b^n \left[\frac{a^{n+1} - b^{n+1}}{b^{n+1}} \times \frac{b}{a-b} \right] = (-1)^n \cancel{b^n} \left[\frac{a^{n+1} - b^{n+1}}{\cancel{b^n} \cancel{b}} \times \frac{\cancel{b}}{a-b} \right] \\
 &= (-1)^n \left[\frac{a^{n+1} - b^{n+1}}{a-b} \right]
 \end{aligned}$$

$$\boxed{Z^{-1} \left[\frac{z^2}{(z+a)(z+b)} \right] = \frac{(-1)^n}{b-a} [b^{n+1} - a^{n+1}]}$$

3. Find $Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right]$ using convolution theorem.

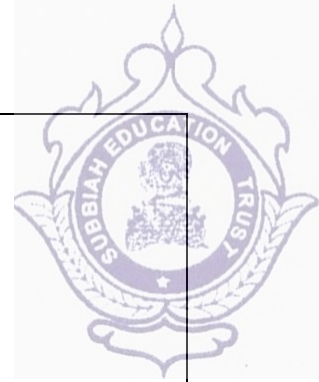
Solution:

Given $Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right] = ?$

By convolution theorem

$$Z^{-1} [F(z) \cdot G(z)] = Z^{-1} [F(z)] * Z^{-1} [G(z)]$$

$$\begin{aligned}
 Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right] &= Z^{-1} \left[\frac{z}{z-a} \cdot \frac{z}{z-b} \right] \\
 &= Z^{-1} \left[\frac{z}{z-a} \right] * Z^{-1} \left[\frac{z}{z-b} \right] \\
 &= (a)^n * (b)^n \\
 &= \sum_{k=0}^n (a)^k (b)^{n-k} \quad \because f(n) * g(n) = \sum_{k=0}^n f(k)g(n-k) \\
 &= \sum_{k=0}^n a^k b^{-k} b^n \\
 &= b^n \sum_{k=0}^n \left(\frac{a}{b}\right)^k \\
 &= b^n \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^3 + \dots + \left(\frac{a}{b}\right)^n \right]
 \end{aligned}$$



$$\begin{aligned}
 &= b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1} \right] = b^n \left[\frac{a^{n+1} - b^{n+1}}{b^{n+1} - 1} \right] = b^n \left[\frac{a^{n+1} - b^{n+1}}{b^{n+1} \left(\frac{a-b}{b}\right)} \right] \\
 &= b^n \left[\frac{a^{n+1} - b^{n+1}}{b^{n+1}} \times \frac{b}{a-b} \right] = (-1)^n \cancel{b^n} \left[\frac{a^{n+1} - b^{n+1}}{\cancel{b^n} \cancel{b}} \times \frac{\cancel{b}}{a-b} \right] \\
 &= \frac{a^{n+1} - b^{n+1}}{a-b}
 \end{aligned}$$

$$\boxed{Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right] = \frac{a^{n+1} - b^{n+1}}{a-b}}$$

4. Using convolution theorem, find $Z^{-1} \left[\frac{8z^2}{(2z-1)(4z+1)} \right]$

Solution:

Given $Z^{-1} \left[\frac{8z^2}{(2z-1)(4z+1)} \right] = ?$

By convolution theorem

$$Z^{-1} [F(z) \cdot G(z)] = Z^{-1} [F(z)] * Z^{-1} [G(z)]$$

$$Z^{-1} \left[\frac{8z^2}{(2z-1)(4z+1)} \right] = Z^{-1} \left[\frac{8z^2}{2\left(z-\frac{1}{2}\right)4\left(z+\frac{1}{4}\right)} \right] = Z^{-1} \left[\frac{z}{\left(z-\frac{1}{2}\right)} \cdot \frac{z}{\left(z-\frac{1}{4}\right)} \right]$$

$$= Z^{-1} \left[\frac{z}{\left(z-\frac{1}{2}\right)} \right] * Z^{-1} \left[\frac{z}{\left(z-\frac{1}{4}\right)} \right]$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$$

$$= \sum_{k=0}^n \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^{n-k} \quad \because f(n) * g(n) = \sum_{k=0}^n f(k)g(n-k)$$

$$= \sum_{k=0}^n \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^n \left(\frac{1}{4}\right)^{-k}$$

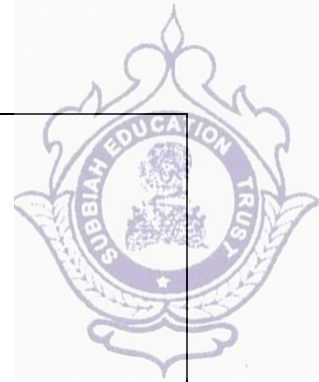
$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^n \left(\frac{1}{2}\right)^k (4)^k = \left(\frac{1}{4}\right)^n \sum_{k=0}^n \left(\frac{4}{2}\right)^k = \left(\frac{1}{4}\right)^n \sum_{k=0}^n (2)^k$$

$$= \left(\frac{1}{4}\right)^n [1+2+2^2+2^3+\dots+2^n]$$

$$= \left(\frac{1}{4}\right)^n \left[\frac{2^{n+1}-1}{2-1} \right] \quad \because 1+a+a^2+a^3+\dots+a^n = \frac{a^{n+1}-1}{a-1}$$

$$\boxed{Z^{-1} \left[\frac{8z^2}{(2z-1)(4z+1)} \right] = \left(\frac{1}{4}\right)^n [2^{n+1}-1]}$$

5. Using convolution theorem find $Z^{-1} \left[\frac{z^2}{(z-1)(z-3)} \right]$

**Solution:**

$$\text{Given } Z^{-1} \left[\frac{z^2}{(z-1)(z-3)} \right] = ?$$

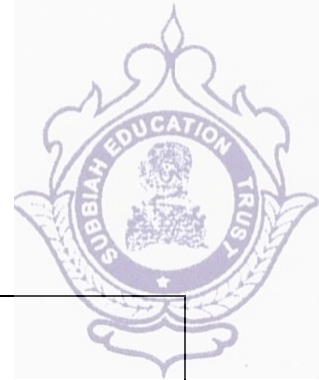
By convolution theorem

$$Z^{-1} [F(z) \cdot G(z)] = Z^{-1} [F(z)] * Z^{-1} [G(z)]$$

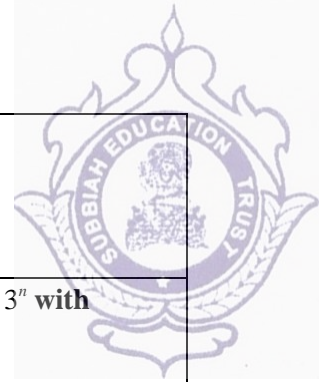
$$\begin{aligned} Z^{-1} \left[\frac{z^2}{(z-1)(z-3)} \right] &= Z^{-1} \left[\frac{z}{z-1} \cdot \frac{z}{z-3} \right] \\ &= Z^{-1} \left[\frac{z}{z-1} \right] * Z^{-1} \left[\frac{z}{z-3} \right] \\ &= (1)^n * (3)^n \\ &= \sum_{k=0}^n (1)^k (3)^{n-k} \quad \because f(n) * g(n) = \sum_{k=0}^n f(k)g(n-k) \\ &= \sum_{k=0}^n 1^k 3^{-k} 3^n \\ &= 3^n \sum_{k=0}^n \left(\frac{1}{3} \right)^k \\ &= 3^n \left[1 + \left(\frac{1}{3} \right) + \left(\frac{1}{3} \right)^2 + \left(\frac{1}{3} \right)^3 + \dots + \left(\frac{1}{3} \right)^n \right] \\ &= 3^n \left[\frac{\left(\frac{1}{3} \right)^{n+1} - 1}{\frac{1}{3} - 1} \right] = 3^n \left[\frac{1^{n+1} - 1}{3^{n+1} - 1} \right] = 3^n \left[\frac{1^{n+1} - 3^{n+1}}{3^{n+1} - 1} \right] \\ &= 3^n \left[\frac{1 - 3^{n+1}}{3^{n+1} - 1} \times \frac{3}{1-3} \right] = \cancel{3^n} \left[\frac{3^{n+1} - 3^{n+1}}{\cancel{3^n} \cancel{3}} \times \frac{\cancel{3}}{-2} \right] \\ &= \frac{-1}{2} [1 - 3^{n+1}] \end{aligned}$$

$$\boxed{Z^{-1} \left[\frac{z^2}{(z-1)(z-3)} \right] = \frac{-1}{2} [1 - 3^{n+1}]}$$

5.4 FORMATION OF DIFFERENCE EQUATION & SOLUTION OF DIFFERENCE EQUATION



	<p>Derive the difference equation from $u_n = a + b3^n$</p> <p>Solution: $u_n = a + b3^n$ -----(1)</p> <p>Replace n by $n+1$ in (1)</p> $u_{n+1} = a + b3^{n+1}$ $u_{n+1} = a + 3b3^n$ -----(2) <p>Replace n by $n+2$ in (1)</p> $u_{n+2} = a + b3^{n+2}$ $u_{n+2} = a + 9b3^n$ -----(3) <p>From (1), (2) and (3)</p> $\begin{vmatrix} u_n & 1 & 1 \\ u_{n+1} & 1 & 3 \\ u_{n+2} & 1 & 9 \end{vmatrix} = 0$ $u_n(9-3) - 1(3u_{n+2} - 9u_{n+1}) + 1(u_{n+1} - u_{n+2}) = 0$ $6u_n - 3u_{n+2} + 9u_{n+1} + u_{n+1} - u_{n+2} = 0$ $-4u_{n+2} + 10u_{n+1} + 6u_n = 0$ $\div(-2) \Rightarrow \boxed{2u_{n+2} - 5u_{n+1} - 3u_n = 0}$
<p>3.</p>	<p>Form the difference equation $y_n = \cos\left(\frac{n\pi}{2}\right)$</p> <p>Solution:</p> <p>Given $y_n = \cos\left(\frac{n\pi}{2}\right)$ -----(1)</p> <p>Replace n by $n+1$ in (1)</p> $y_{n+1} = \cos\left(\frac{(n+1)\pi}{2}\right) = \cos\left(\frac{\pi}{2} + \frac{n\pi}{2}\right) = -\sin\left(\frac{n\pi}{2}\right)$ -----(2) <p>Replace n by $n+2$ in (1)</p> $y_{n+2} = \cos\left(\frac{(n+2)\pi}{2}\right) = \cos\left(\frac{2\pi}{2} + \frac{n\pi}{2}\right)$ $y_{n+2} = \cos\left(\pi + \frac{n\pi}{2}\right) = -\cos\left(\frac{n\pi}{2}\right)$ <p>$y_{n+2} = -y_n$ from (1)</p> $\Rightarrow \boxed{y_{n+2} + y_n = 0}$
<p>Solutions of difference equation using Z-Transforms.</p> <p>1. $Z[y_n] = Z[y(n)] = y(z)$</p>	



2. $Z[y_{n+1}] = Z[y(n+1)] = zy(z) - zy(0)$
3. $Z[y_{n+2}] = Z[y(n+2)] = z^2y(z) - z^2y(0) - zy(1)$
4. $Z[y_{n+3}] = Z[y(n+3)] = z^3y(z) - z^3y(0) - z^2y(1) - zy(2)$

1. **Solve using Z-transforms technique the difference equation $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$ with**

$y_0 = 0, y_1 = 1.$

Solution:

$$y_{n+2} + 4y_{n+1} + 3y_n = 3^n .$$

Taking Z-transform on both sides

$$Z[y_{n+2}] + 4Z[y_{n+1}] + 3Z[y_n] = Z[3^n]$$

$$\left[z^2y(z) - z^2y(0) - zy(1) \right] + 4\left[zy(z) - zy(0) \right] + 3y(z) = \frac{z}{z-3}$$

Given $y_0 = y(0) = 0, y_1 = y(1) = 1$

$$z^2y(z) - z + 4zy(z) + 3y(z) = \frac{z}{z-3}$$

$$(z^2 + 4z + 3)y(z) = \frac{z}{z-3} + z$$

$$(z^2 + 4z + 3)y(z) = \frac{z + z^2 - 3z}{z-3}$$

$$y(z) = \frac{z^2 - 2z}{(z-3)(z^2 + 4z + 3)}$$

$$y(z) = \frac{z(z-2)}{(z-3)(z+1)(z+3)}$$

By Partial Fraction,

$$\frac{y(z)}{z} = \frac{(z-2)}{(z-3)(z+1)(z+3)} \text{ ----- (1)}$$

Now $\frac{(z-2)}{(z-3)(z+1)(z+3)} = \frac{A}{(z-3)} + \frac{B}{(z+1)} + \frac{C}{(z+3)}$

$$z-2 = A(z+1)(z+3) + B(z-3)(z+3) + C(z+1)(z-3)$$

Put z=3 $\Rightarrow 1 = 24A \Rightarrow A = \frac{1}{24}$

Put z=-1 $\Rightarrow -3 = -8B \Rightarrow B = \frac{3}{8}$

Put z=-3 $\Rightarrow -5 = 12C \Rightarrow C = \frac{-5}{12}$

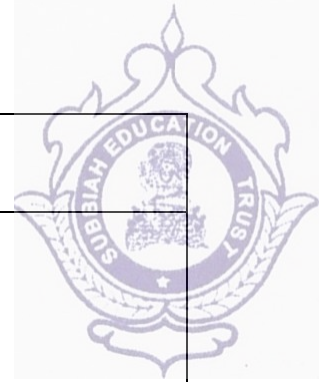
$$\frac{(z-2)}{(z-3)(z+1)(z+3)} = \frac{1/24}{(z-3)} + \frac{3/8}{(z+1)} + \frac{-5/12}{(z+3)}$$

(1) $\Rightarrow \frac{y(z)}{z} = \frac{1/24}{(z-3)} + \frac{3/8}{(z+1)} + \frac{-5/12}{(z+3)}$

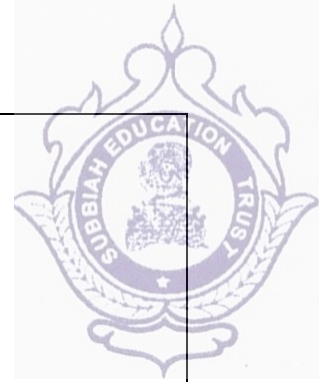
$$y(z) = \frac{1}{24} \frac{z}{(z-3)} + \frac{3}{8} \frac{z}{(z+1)} - \frac{5}{12} \frac{z}{(z+3)}$$

Taking Z^{-1} on both sides

$$Z^{-1}[y(z)] = \frac{1}{24} Z^{-1}\left[\frac{z}{z-3}\right] + \frac{3}{8} Z^{-1}\left[\frac{z}{z+1}\right] - \frac{5}{12} Z^{-1}\left[\frac{z}{z+3}\right]$$



	$y(n) = \frac{1}{24}(3)^n + \frac{3}{8}(-1)^n - \frac{5}{12}(-3)^n \qquad \because Z^{-1} \left[\frac{z}{z-a} \right] = a^n$
<p>2.</p>	<p>Solve $y_{n+2} - 3y_{n+1} - 10y_n = 0$, given $y_0 = 1, y_1 = 0$. Solution: $y_{n+2} - 3y_{n+1} - 10y_n = 0$. Taking Z-transform on both sides $Z[y_{n+2}] - 3Z[y_{n+1}] - 10Z[y_n] = Z[0]$ $[z^2 y(z) - z^2 y(0) - zy(1)] - 3[zy(z) - zy(0)] - 10y(z) = 0$ Given $y_0 = y(0) = 1, y_1 = y(1) = 0$ $z^2 y(z) - z^2 - 3zy(z) + 3z - 10y(z) = 0$ $(z^2 - 3z - 10)y(z) = z^2 - 3z$ $y(z) = \frac{z^2 - 3z}{(z^2 - 3z - 10)}$ $y(z) = \frac{z(z - 3)}{(z + 2)(z - 5)}$ By Partial Fraction, $\frac{y(z)}{z} = \frac{(z - 3)}{(z + 2)(z - 5)} \text{ ----- (1)}$ Now $\frac{(z - 3)}{(z + 2)(z - 5)} = \frac{A}{(z + 2)} + \frac{B}{(z - 5)}$ $z - 3 = A(z - 5) + B(z + 2)$ Put $z = -2 \Rightarrow -5 = -7A \Rightarrow A = \frac{5}{7}$ Put $z = 5 \Rightarrow 2 = 7B \Rightarrow B = \frac{2}{7}$ $\frac{(z - 3)}{(z + 2)(z - 5)} = \frac{\frac{5}{7}}{(z + 2)} + \frac{\frac{2}{7}}{(z - 5)}$ $(1) \Rightarrow \frac{y(z)}{z} = \frac{5}{7} \frac{1}{z + 2} + \frac{2}{7} \frac{1}{z - 5}$ $y(z) = \frac{5}{7} \frac{z}{z + 2} + \frac{2}{7} \frac{z}{z - 5}$ Taking Z^{-1} on both sides $Z^{-1}[y(z)] = \frac{5}{7} Z^{-1} \left[\frac{z}{z + 2} \right] + \frac{2}{7} Z^{-1} \left[\frac{z}{z - 5} \right]$ $y(n) = \frac{5}{7}(-2)^n - \frac{2}{7}5^n \qquad \because Z^{-1} \left[\frac{z}{z-a} \right] = a^n$</p>
<p>3.</p>	<p>Solve the equation $y(n + 3) - 3y(n + 1) + 2y(n) = 0$ given that $y(0) = 4, y(1) = 0$ and $y(2) = 8$. Solution: $Z[y(n + 3)] - 3Z[y(n + 1)] + 2Z[y(n)] = Z[0]$ $[z^3 y(z) - z^3 y(0) - z^2 y(1) - zy(2)] - 3[zy(z) - zy(0)] + 2y(z) = 0$ Given that $y(0) = 4, y(1) = 0$</p>



$$z^3 y(z) - 4z^3 - 8z - 3zy(z) + 12z + 2y(z) = 0$$

$$[z^3 - 3z + 2]y(z) = 4z^3 - 4z$$

$$y(z) = \frac{4z^3 - 4z}{z^3 - 3z + 2}$$

$$y(z) = \frac{4z(z^2 - 1)}{(z-1)^2(z+2)}$$

$$y(z) = \frac{4z(z-1)(z+1)}{(z-1)^2(z+2)} \quad \because a^2 - b^2 = (a+b)(a-b)$$

$$y(z) = \frac{4z(z+1)}{(z-1)(z+2)}$$

By Partial Fraction

$$\frac{y(z)}{z} = \frac{4(z+1)}{(z-1)(z+2)} \text{ ----- (1)}$$

$$\frac{4(z+1)}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$$

$$4(z+1) = A(z+2) + B(z-1)$$

Put $z = 1 \Rightarrow 8 = 3A \Rightarrow A = \frac{8}{3}$

Put $z = -2 \Rightarrow -4 = -3B \Rightarrow B = \frac{4}{3}$

$$\frac{y(z)}{z} = \frac{8/3}{z-1} + \frac{4/3}{z+2}$$

$$Z^{-1}[y(z)] = \frac{8}{3}Z^{-1}\left[\frac{z}{z-1}\right] + \frac{4}{3}Z^{-1}\left[\frac{z}{z+2}\right]$$

$$\boxed{y(n) = \frac{8}{3} + \frac{4}{3}(-2)^n} \quad \because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

4. Using Z-transform solve $y(n) + 3y(n-1) - 4y(n-2) = 0, n \geq 2$ given that

$y(0) = 3$ and $y(1) = -2$

Solution:

Given $y(n) + 3y(n-1) - 4y(n-2) = 0, n \geq 2$

Replace n by $n+2$, we get

$$y(n+2) + 3y(n+1) - 4y(n) = 0$$

Taking Z transforms on both sides

$$Z[y(n+2)] + 3Z[y(n+1)] - 4Z[y(n)] = Z[0]$$

$$[z^2 y(z) - z^2 y(0) - zy(1)] + 3[zy(z) - zy(0)] - 4y(z) = 0$$

Given that $y(0) = 3$ and $y(1) = -2$

$$[z^2 y(z) - 3z^2 + 2z] + 3[zy(z) - 3z] - 4y(z) = 0$$

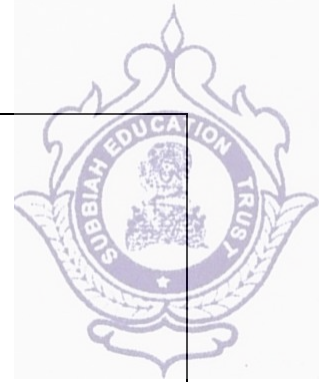
$$[z^2 + 3z - 4]y(z) - 3z^2 + 2z - 9z = 0$$

$$[z^2 + 3z - 4]y(z) = 3z^2 + 7z$$

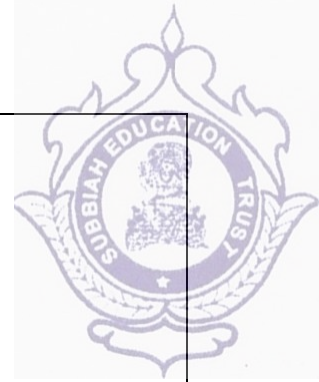
$$y(z) = \frac{3z^2 + 7z}{z^2 + 3z - 4}$$

By Partial Fraction

$$\frac{y(z)}{z} = \frac{3z+7}{z^2+3z-4} = \frac{3z+7}{(z+4)(z-1)}$$



	<p>Now, $\frac{3z+7}{(z+4)(z-1)} = \frac{A}{z+4} + \frac{B}{z-1}$</p> <p>$3z+7 = A(z-1) + B(z+4)$</p> <p>Put $z=1 \Rightarrow 10 = 5B \Rightarrow B=2$</p> <p>Put $z=-4 \Rightarrow -5 = -5A \Rightarrow A=1$</p> <p>$\frac{y(z)}{z} = \frac{1}{z+4} + \frac{2}{z-1}$</p> <p>$y(z) = \frac{z}{z+4} + 2\frac{z}{z-1}$</p> <p>$Z^{-1}[y(z)] = Z^{-1}\left[\frac{z}{z+4}\right] + 2Z^{-1}\left[\frac{z}{z-1}\right]$</p> <div style="display: flex; justify-content: space-between; align-items: center;"> <div style="border: 1px solid black; padding: 2px;">$y(n) = (-4)^n + 2(1)^n = 2 + (-4)^n$</div> <div>$\therefore Z^{-1}\left[\frac{z}{z-a}\right] = a^n$</div> </div>
5.	<p>Solve using Z-transforms technique the difference equation $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$ with $u_0 = u_1 = 0$.</p> <p>Solution:</p> <p>$u_{n+2} + 6u_{n+1} + 9u_n = 2^n$</p> <p>Assume $u=y$</p> <p>$y_{n+2} + 6y_{n+1} + 9y_n = 2^n ; y_0 = y_1 = 0$</p> <p>Taking Z-transform on both sides</p> <p>$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] = Z[2^n]$</p> <p>$[z^2y(z) - z^2y(0) - zy(1)] + 6[zy(z) - zy(0)] + 9y(z) = \frac{z}{z-2}$</p> <p>Given $y_0 = y(0) = 0 ; y_1 = y(1) = 0$</p> <p>$z^2y(z) + 6zy(z) + 9y(z) = \frac{z}{z-2}$</p> <p>$(z^2 + 6z + 9)y(z) = \frac{z}{z-2}$</p> <p>$y(z) = \frac{z}{(z-2)(z^2 + 6z + 9)}$</p> <p>$y(z) = \frac{z}{(z-2)(z+3)^2}$</p> <p>By Partial Fraction,</p> <p>$\frac{y(z)}{z} = \frac{1}{(z-2)(z+3)^2}$ ----- (1)</p> <p>Now $\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$</p> <p>$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$</p> <p>Put $z=2 \Rightarrow 1 = 25A \Rightarrow A = \frac{1}{25}$</p> <p>Put $z=-3 \Rightarrow 1 = -5C \Rightarrow C = \frac{-1}{5}$</p> <p>Equating co-efft. of z^2 on both sides $\Rightarrow A+B=0 \Rightarrow B=-A \Rightarrow B = -\frac{1}{25}$</p>



$$\frac{y(z)}{z} = \frac{1}{25} \frac{1}{(z-2)} + \frac{-1}{25} \frac{1}{(z+3)} + \frac{-1}{5} \frac{1}{(z+3)^2}$$

Taking Z^{-1} on both sides

$$Z^{-1}[y(z)] = \frac{1}{25} Z^{-1} \left[\frac{z}{z-2} \right] - \frac{1}{25} Z^{-1} \left[\frac{z}{z+3} \right] - \frac{1}{5} Z^{-1} \left[\frac{z}{(z+3)^2} \right]$$

$$\boxed{y(n) = \frac{1}{25} (2)^n - \frac{1}{25} (-3)^n - \frac{1}{5} n(-3)^{n-1}} \quad \because Z^{-1} \left[\frac{z}{(z-a)^2} \right] = na^{n-1} \text{ \& } Z^{-1} \left[\frac{z}{z-a} \right] = a^n$$

$$\boxed{u(n) = \frac{1}{25} (2)^n - \frac{1}{25} (-3)^n - \frac{1}{5} n(-3)^{n-1}} \quad \because u = y$$

6. Using Z-transform method solve $y(k+2) + y(k) = 2$ given that $y_0 = y_1 = 0$.

Solution:

Given $y(k+2) + y(k) = 2$; $y_0 = y_1 = 0$.

Assume $k=n$

$$y(n+2) + y(n) = 2$$

Taking Z-transform on both sides

$$Z[y(n+2)] + Z[y(n)] = 2Z[1]$$

$$\left[z^2 y(z) - z^2 y(0) - zy(1) \right] + y(z) = 2 \frac{z}{z-1}$$

Given that $y_0 = y_1 = 0$.

$$(z^2 + 1)y(z) = \frac{2z}{z-1}$$

$$y(z) = \frac{2z}{(z-1)(z^2+1)}$$

$$\frac{y(z)}{z} = \frac{2}{(z-1)(z^2+1)} \text{ ----- (1)}$$

By partial fraction

$$\text{Now, } \frac{2}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{B}{z^2+1} + \frac{Cz}{z^2+1}$$

$$2 = A(z^2+1) + B(z-1) + Cz(z-1)$$

$$\text{Put } z=1 \Rightarrow 2 = 2A \Rightarrow A=1$$

$$\text{Put } z=0 \Rightarrow 2 = A - B \Rightarrow B = A - 2 \Rightarrow B = -1$$

$$\text{Equating co-efft. of } z^2 \text{ on both sides } \Rightarrow 0 = A + C \Rightarrow C = -A \Rightarrow C = -1$$

$$(1) \Rightarrow \frac{y(z)}{z} = \frac{1}{z-1} + \frac{-1}{z^2+1} + \frac{-z}{z^2+1}$$

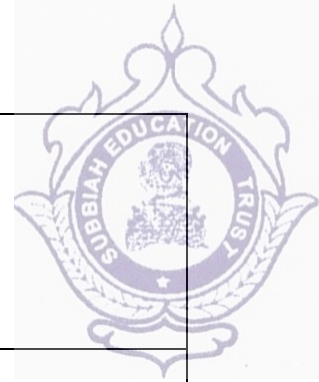
$$y(z) = \frac{z}{z-1} - \frac{z}{z^2+1} - \frac{z^2}{z^2+1}$$

Taking Z^{-1} on both sides

$$Z^{-1}[y(z)] = Z^{-1} \left[\frac{z}{z-1} \right] - Z^{-1} \left[\frac{z}{z^2+1} \right] - Z^{-1} \left[\frac{z^2}{z^2+1} \right]$$

$$y(n) = (1)^n - 1^n \sin \frac{n\pi}{2} - 1^n \cos \frac{n\pi}{2}$$

$$\boxed{y(n) = 1 - \sin \frac{n\pi}{2} - \cos \frac{n\pi}{2}} \quad \boxed{y(k) = 1 - \sin \frac{k\pi}{2} - \cos \frac{k\pi}{2}}$$



5.5 Z-Transforms and its Properties

1. Find $Z[\cos n\theta]$, $Z[\sin n\theta]$ and hence find i) $Z\left[\cos \frac{n\pi}{2}\right]$, ii) $Z\left[\sin \frac{n\pi}{2}\right]$
 iii) $Z[r^n \cos n\theta]$ iv) $Z[r^n \sin n\theta]$

Solution:

We know that $e^{in\theta} = \cos n\theta + i \sin n\theta$

$\cos n\theta = \text{real part of } e^{in\theta}$ & $\sin n\theta = \text{imaginary part of } e^{in\theta}$

and $Z[a^n] = \frac{z}{z-a}$

$$Z[e^{in\theta}] = Z[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}}$$

$$= \frac{z}{z-(\cos \theta + i \sin \theta)}$$

$$= \frac{z}{(z-\cos \theta) - i \sin \theta} \times \frac{(z-\cos \theta) + i \sin \theta}{(z-\cos \theta) + i \sin \theta}$$

$$Z[e^{in\theta}] = \frac{z(z-\cos \theta) + i \sin \theta}{(z-\cos \theta)^2 - i^2 \sin^2 \theta} \quad \because (a+b)(a-b) = a^2 - b^2$$

$$Z[\cos n\theta + i \sin n\theta] = \frac{z(z-\cos \theta) + iz \sin \theta}{z^2 - 2z \cos \theta + \cos^2 \theta + \sin^2 \theta} \quad \because i^2 = -1$$

$$Z[\cos n\theta] + iZ[\sin n\theta] = \frac{z(z-\cos \theta)}{z^2 - 2z \cos \theta + 1} + i \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1} \quad \because \cos^2 \theta + \sin^2 \theta = 1$$

Equating co-efft. Of real and img parts on both sides

$$Z[\cos n\theta] = \frac{z(z-\cos \theta)}{z^2 - 2z \cos \theta + 1} \quad ; \quad Z[\sin n\theta] = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$$

Deduction:

We know that

$$Z[\cos n\theta] = \frac{z(z-\cos \theta)}{z^2 - 2z \cos \theta + 1}$$

$$\text{i) } Z\left[\cos \frac{n\pi}{2}\right] = Z[\cos n\theta]_{\theta \rightarrow \frac{\pi}{2}} = \frac{z\left(z - \cos \frac{\pi}{2}\right)}{z^2 - 2z \cos \frac{\pi}{2} + 1}$$

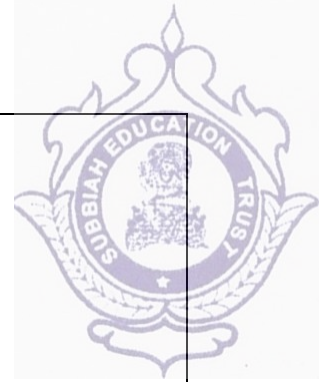
$$Z\left[\cos \frac{n\pi}{2}\right] = \frac{z^2}{z^2 + 1} \quad \because \cos \frac{\pi}{2} = 0$$

$$Z[\sin n\theta] = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$$

$$\text{ii) } Z\left[\sin \frac{n\pi}{2}\right] = Z[\sin n\theta]_{\theta \rightarrow \frac{\pi}{2}} = \frac{z \sin \frac{\pi}{2}}{z^2 - 2z \cos \frac{\pi}{2} + 1}$$

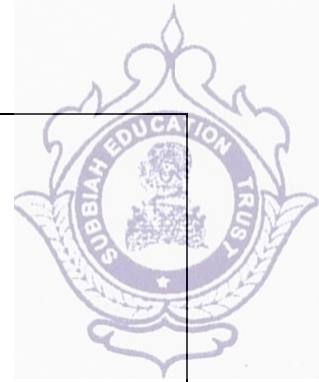
$$\therefore Z\left[\sin \frac{n\pi}{2}\right] = \frac{z}{z^2 + 1} \quad \because \cos \frac{\pi}{2} = 0 \text{ \& } \sin \frac{\pi}{2} = 1$$

We know that



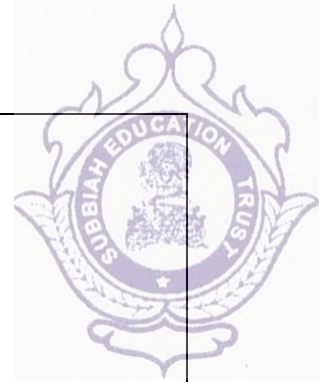
	$Z[a^n f(n)] = Z[f(n)]_{z \rightarrow \frac{z}{a}}$ <p>iii) $Z[r^n \cos n\theta] = [Z[\cos n\theta]]_{z \rightarrow \frac{z}{r}}$</p> $= \left[\frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1} \right]_{z \rightarrow \frac{z}{r}}$ $= \left[\frac{\frac{z}{r} \left(\frac{z}{r} - \cos \theta \right)}{\frac{z}{r^2} - \frac{2z}{r} \cos \theta + 1} \right]$ $= \frac{\frac{z}{r} \left(\frac{z - r \cos \theta}{r} \right)}{z^2 - 2zr \cos \theta + r^2}$ $Z[r^n \cos n\theta] = \frac{z(z - r \cos \theta)}{z^2 - 2zr \cos \theta + r^2}$ <p>iv) $Z[r^n \sin n\theta] = [Z\{\sin n\theta\}]_{z \rightarrow \frac{z}{r}} = \frac{\frac{z}{r} \sin \theta}{\frac{z}{r^2} - 2\frac{z}{r} \cos \theta + r^2} = \frac{\frac{z}{r} \sin \theta}{z^2 - 2zr \cos \theta + r^2}$</p> $Z[r^n \sin n\theta] = \frac{zr \sin \theta}{z^2 - 2zr \cos \theta + r^2}$
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<p>2.</p>	<p>Find the Z-transform of $\frac{1}{n(n+1)}$, for $n \geq 1$</p> <p>Solution</p> $Z\left[\frac{1}{n(n+1)}\right] = ?$ <p>By partial Fraction:</p> $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$ $1 = A(n+1) + Bn$ <p>Put $n = -1$; $1 = -B \Rightarrow B = -1$</p> <p>Put $n = 0$; $A = 1$</p> $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ $Z\left[\frac{1}{n(n+1)}\right] = Z\left[\frac{1}{n}\right] - Z\left[\frac{1}{n+1}\right] \text{-----(1)}$ <p>Now, we know that</p> $Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$ $Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{z}\right)^n \quad \because n > 0$ $= \frac{1}{z} + \frac{1}{2} \left(\frac{1}{z}\right)^2 + \frac{1}{3} \left(\frac{1}{z}\right)^3 + \dots$
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	$= x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \quad \text{here } \frac{1}{z} = x$ $= -\log(1-x)$ $Z\left[\frac{1}{n}\right] = -\log\left(1 - \frac{1}{z}\right) = -\log\left(\frac{z-1}{z}\right) = \log\left(\frac{z}{z-1}\right)$ $Z\left[\frac{1}{n}\right] = \log\left(\frac{z}{z-1}\right)$ $Z\left(\frac{1}{n+1}\right) = \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\frac{1}{z}\right)^n$ $= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \dots$ $= z\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \dots\right]$ $= z\left[-\log\left(1 - \frac{1}{z}\right)\right] = -z \log\left(\frac{z-1}{z}\right)$ $Z\left(\frac{1}{n+1}\right) = z \log\left(\frac{z}{z-1}\right)$ $(1) \Rightarrow Z\left[\frac{1}{n(n+1)}\right] = \log\left(\frac{z}{z-1}\right) + z \log\left(\frac{z}{z-1}\right)$ $\therefore Z\left[\frac{1}{n(n+1)}\right] = (z+1) \log\left(\frac{z}{z-1}\right)$
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<p>3.</p>	<p>Find $Z[n(n-1)(n-2)]$.</p> <p>Solution:</p> $Z[n(n-1)(n-2)] = Z[(n^2 - n)(n-2)] = Z[n^3 - 2n^2 - n^2 + 2n] = Z[n^3 - 3n^2 + 2n]$ $Z[n(n-1)(n-2)] = Z[n^3] - 3Z[n^2] + 2Z[n] \quad \text{-----(1)}$ <p>We know that</p> $Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$ $Z[n] = \sum_{n=0}^{\infty} n\left(\frac{1}{z}\right)^n$ $= 0 + 1\left(\frac{1}{z}\right)^1 + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + \dots$ $= x + 2x^2 + 3x^3 + \dots$ $= x(1 + 2x + 3x^2 + \dots) = x(1-x)^{-2} = \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2}$ $= \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2} = \frac{1}{z}\left(\frac{z}{z-1}\right)^2 = \frac{1}{z}\left(\frac{z^2}{(z-1)^2}\right)$ $Z[n] = \frac{z}{(z-1)^2}$ <p>We know that $Z[nf(n)] = -z \frac{d}{dz} \{Z[f(n)]\}$</p>
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$$\begin{aligned} Z[n^2] &= -z \frac{d}{dz} \{Z[n]\} \\ &= -z \frac{d}{dz} \left\{ \frac{z}{(z-1)^2} \right\} \\ &= -z \left\{ \frac{(z-1)^2(1) - z[2(z-1)]}{(z-1)^4} \right\} \\ &= -z \left\{ \frac{(z-1)(z-1-2z)}{(z-1)^4} \right\} \\ &= -z \left\{ \frac{-1-z}{(z-1)^3} \right\} \end{aligned}$$

$$Z[n^2] = \frac{z+z^2}{(z-1)^3}$$

$$Z[n^3] = Z[nn^2] = -z \frac{d}{dz} \{Z[n^2]\}$$

$$\begin{aligned} &= -z \frac{d}{dz} \left\{ \frac{z+z^2}{(z-1)^3} \right\} \\ &= -z \left[\frac{(z-1)^3(2z+1) - (z^2+z)3(z-1)^2(1-0)}{(z-1)^6} \right] \\ &= -z \left[\frac{(z-1)^2 [(z-1)(2z+1) - 3(z^2+z)]}{(z-1)^6} \right] \\ &= -z \left[\frac{2z^2 - 2z + z - 1 - 3z^2 - 3z}{(z-1)^4} \right] \\ &= -z \left[\frac{-z^2 - 4z - 1}{(z-1)^4} \right] \end{aligned}$$

$$Z[n^3] = \frac{z(z^2+4z+1)}{(z-1)^4}$$

$$(1) \Rightarrow Z[n(n-1)(n-2)] = \frac{z(z^2+4z+1)}{(z-1)^4} - 3 \frac{z+z^2}{(z-1)^3} + 2 \frac{z}{(z-1)^2}$$

4. If $U(z) = \frac{2z^2+5z+14}{(z-1)^4}$, evaluate u_2 and u_3 .

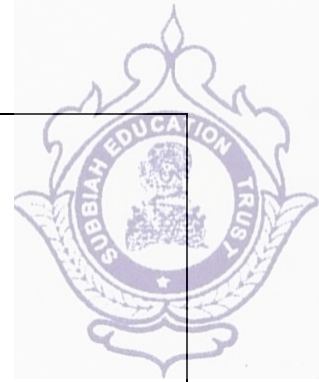
Solution:

Given $U(z) = F(z) = \frac{2z^2+5z+14}{(z-1)^4}$

We know that

$$u_0 = f(0) = \lim_{z \rightarrow \infty} F(z) = \lim_{z \rightarrow \infty} \frac{2z^2+5z+14}{(z-1)^4} = \lim_{z \rightarrow \infty} \frac{z^2 \left(2 + \frac{5}{z} + \frac{14}{z^2} \right)}{z^4 \left(1 - \frac{1}{z} \right)^2}$$

$$u_0 = f(0) = 0 \because \frac{1}{\infty} = 0$$



$$\begin{aligned}
 u_1 = f(1) &= \lim_{z \rightarrow \infty} [zF(z) - zf(0)] \\
 &= \lim_{z \rightarrow \infty} \left[\frac{z(2z^2 + 5z + 14)}{(z-1)^4} - z(0) \right] \\
 &= \lim_{z \rightarrow \infty} \left[\frac{z^3 \left(2 + \frac{5}{z} + \frac{14}{z^2} \right)}{z^4 \left(1 - \frac{1}{z} \right)^4} - 0 \right]
 \end{aligned}$$

$$u_1 = f(1) = 0 \quad \because \frac{1}{\infty} = 0$$

$$\begin{aligned}
 u_2 = f(2) &= \lim_{z \rightarrow \infty} [z^2 F(z) - z^2 f(0) - zf(1)] \\
 &= \lim_{z \rightarrow \infty} \left[\frac{z^2(2z^2 + 5z + 14)}{(z-1)^4} - z^2(0) - z(0) \right] \\
 &= \lim_{z \rightarrow \infty} \left[\frac{z^4 \left(2 + \frac{5}{z} + \frac{14}{z^2} \right)}{z^4 \left(1 - \frac{1}{z} \right)^4} \right] = \frac{2+0+0}{(1-0)^4} = 2
 \end{aligned}$$

$$u_2 = f(2) = 2$$

$$\begin{aligned}
 u_3 = f(3) &= \lim_{z \rightarrow \infty} [z^3 F(z) - z^3 f(0) - z^2 f(1) - zf(2)] \\
 &= \lim_{z \rightarrow \infty} \left[\frac{z^3(2z^2 + 5z + 14)}{(z-1)^4} - z^3(0) - z^2(0) - z(2) \right] \\
 &= \lim_{z \rightarrow \infty} \left[\frac{z^3(2z^2 + 5z + 14)}{(z-1)^4} - 2z \right] \\
 &= \lim_{z \rightarrow \infty} z^3 \left(\frac{(2z^2 + 5z + 14)}{(z-1)^4} - \frac{2}{z^2} \right) \\
 &= \lim_{z \rightarrow \infty} z^3 \left(\frac{z^2(2z^2 + 5z + 14) - 2(z-1)^4}{z^2(z-1)^4} \right) \because (a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^3 \\
 &= \lim_{z \rightarrow \infty} z^3 \left(\frac{(2z^4 + 5z^3 + 14z^2) - 2(z^4 - 4z^3 + 6z^2 - 4z + 1)}{z^2(z-1)^4} \right) \\
 &= \lim_{z \rightarrow \infty} z^3 \left(\frac{2z^4 + 5z^3 + 14z^2 - 2z^4 + 8z^3 - 12z^2 + 8z - 2}{z^2(z-1)^4} \right) \\
 &= \lim_{z \rightarrow \infty} z^3 \left(\frac{13z^3 + 2z^2 + 8z - 2}{z^2(z-1)^4} \right) \\
 &= \lim_{z \rightarrow \infty} \frac{z^6 \left(13 + \frac{2}{z} + \frac{8}{z^2} - \frac{2}{z^3} \right)}{z^6 \left(1 - \frac{1}{z} \right)^4} = \lim_{z \rightarrow \infty} \frac{\left(13 + \frac{2}{z} + \frac{8}{z^2} - \frac{2}{z^3} \right)}{\left(1 - \frac{1}{z} \right)^4} = \frac{13+0+0-0}{(1-0)^4}
 \end{aligned}$$

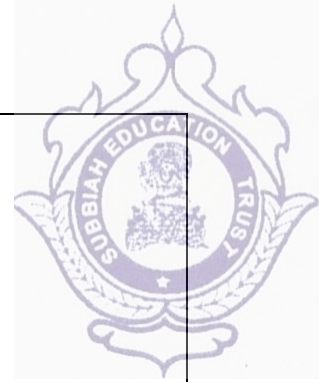
$$u_3 = f(3) = 13$$

5. State and prove initial and final value theorem of Z-transform.

Initial value theorem:

If $Z[f(n)] = F(z)$ **then** $f(0) = \lim_{z \rightarrow \infty} F(z)$

Proof:



We know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$$

$$\begin{aligned} \lim_{z \rightarrow \infty} F(z) &= \lim_{z \rightarrow \infty} \sum_{n=0}^{\infty} f(n) \left(\frac{1}{z}\right)^n \\ &= \lim_{z \rightarrow \infty} \left[f(0) \left(\frac{1}{z}\right)^0 + f(1) \left(\frac{1}{z}\right)^1 + f(2) \left(\frac{1}{z}\right)^2 + \dots \right] \end{aligned}$$

$$\lim_{z \rightarrow \infty} F(z) = f(0) \quad \because \frac{1}{\infty} = 0$$

Final value theorem:

If $Z[f(n)] = F(z)$ **then** $\lim_{n \rightarrow \infty} f(n) = \lim_{z \rightarrow 1} (z-1)F(z)$

Proof:

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n} \quad \text{----- (1)}$$

$$Z[f(n+1)] = \sum_{n=0}^{\infty} f(n+1)z^{-n} \quad \text{----- (2)}$$

$$(1) - (2) \Rightarrow$$

$$Z[f(n+1)] - Z[f(n)] = \sum_{n=0}^{\infty} f(n+1)z^{-n} - \sum_{n=0}^{\infty} f(n)z^{-n}$$

$$[zF(z) - zf(0)] - F(z) = \sum_{n=0}^{\infty} [f(n+1) - f(n)]z^{-n}$$

$$\lim_{z \rightarrow 1} [(z-1)F(z) - zf(0)] = \lim_{z \rightarrow 1} \sum_{n=0}^{\infty} [f(n+1) - f(n)]z^{-n}$$

$$\lim_{z \rightarrow 1} [(z-1)F(z)] - f(0) = \sum_{n=0}^{\infty} [f(n+1) - f(n)]$$

$$\lim_{z \rightarrow 1} [(z-1)F(z)] - f(0) = [f(1) - f(0)] + [f(2) - f(1)] + \dots + [f(n+1) - f(n)] + \dots \infty$$

$$\lim_{z \rightarrow 1} [(z-1)F(z)] - f(0) = -f(0) + f(n+1) + \dots \infty$$

$$\lim_{z \rightarrow 1} [(z-1)F(z)] = \lim_{n \rightarrow \infty} f(n) \quad \because f(n+1) = f(n) \text{ when } n \rightarrow \infty$$

Hence proved